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Strong Asymptotics Of Pade Polynomials

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STRONG ASYMPTOTICS OF PADÉ POLYNOMIALS

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ABSTRACT

New results about the strong asymptotic behaviour of diagonal Padé polynomials of high degree are obtained for certain functions with branch points. The method, a modification of a previous approach, uses a singular integral equation for the remainder function restricted to a preferred set. New techniques are developed to analyze three cases

1) a branch point not of square-root type
2) a case where the preferred set contains three intersecting arcs
3) a case where not all zeros of the polynomials approach the preferred set.

The first two cases have not previously been treated. The method involves approximating the kernel of the integral equation so that the resulting equation may be solved analytically. The third case has been treated before by a different method, but it is important to show that the new method can handle this case.

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To Dr. and Mrs. Nuttall

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CHAPTER 1: PADÉ APPROXIMANTS AND PADÉ POLYNOMIALS

The aim of this thesis is to develop new methods of proving results about the strong asymptotics of Padé polynomials. In order to explain these results, it is necessary to first introduce the appropriate background material and notation. A brief outline of some basic properties of Padé approximants and the Padé polynomials from which they are formed is found in Chapter 1. In Chapter 2, some previous work on the convergence of Padé approximants and the asymptotics of Padé polynomials for special cases is reviewed. Fundamental to the present work are the ideas of Nuttall and Stahl on the asymptotics of Padé polynomials associated with functions having branch points, and an introduction to this topic is given in Chapter 3. The reader who is familiar with all this material may immediately turn to Chapter 4 to find an outline of the results obtained in this thesis. Chapter 5 develops the form of the integral equation which is used to obtain the asymptotic results. The following three chapters apply the equation to representative examples of functions that illustrate how to overcome three types of difficulty that can arise in the area of study. Chapter 9 concludes the thesis and describes avenues for further research suggested by the results.

1.1 Introduction

It has been almost a century since the French mathematician H. Padé first published his paper concerning the approximate representation of a function by rational fraction [33]. However, the advantages of Padé method were not really recognized until the mid-1950's. After a paper published by D. Shanks [38] in 1955, many physicists and applied mathematicians started to become interested in research about and applications of Padé methods. Especially, in the early 1960's G. Baker and J. Gammel introduced some important and fundamental ideas about this method, including the importance of Padé approximants as a systematic method of extracting more information from power series expansions, the invariance theorem and the unitary theorem [2], [7]. The convergence theory of Padé approximants and the closely related problem of the asymptotics of Padé polynomials were mainly developed by J. Nuttall from 1970 onwards and by H. Stahl in 1980's.

The Padé method has grown rapidly as an important and standard technique in scientific research, and wide application in the fields of physics, numerical analysis, computation, fluid mechanics, and even theoretical chemistry and engineering has occurred.

1.2 Definition of Padé Approximants and Padé Polynomials

The Padé approximant aims to represent a function by the ratio of two polynomials. The coefficients occurring in the polynomials are determined by the coefficients in the Taylor series expansion of the function being approximated. Thus, suppose that we are given a power series

$$F(x) = c_0 + c_1x + c_2x^2 + \ldots, \qquad x \to 0. \tag{1.2.1}$$

A Padé approximant to $F(x)$ is a rational fraction

$$[m/n] = \frac{Q(x)}{P(x)} = \frac{a_0 + a_1 x + \dots + a_m x^m}{b_0 + b_1 x + \dots + b_n x^n}, \tag{1.2.2}$$

which has a Maclaurin expansion which agrees with (1.2.1) as far as possible, where $P(x)$ is a polynomial of degree at most m and $Q(x)$ is a polynomial of degree at most n.

A more satisfactory definition for our purposes, called the Frobenius definition, is given as follows.

Definition 1.1 *If polynomials* $Q_{m,n}(x)$, $P_{m,n}(x)$ *of degrees at most* m, n *respectively are found from*

$$Q_{m,n}(x) - P_{m,n}(x)\ F(x) = O(x^{m+n+1}), \qquad x \to 0, \tag{1.2.3}$$

then the m, n *Padé approximant to* $F(x)$ *is written as*

$$[m/n] = Q_{m,n}(x) \ / \ P_{m\,n}(x). \tag{1.2.4}$$

The polynomials $Q_{m,n}$, $P_{m,n}$ *always exist but may not be unique. However, the ratio* $Q_{m,n}$ / $P_{m,n}$ *is unique.*

It is in some ways more convenient to expand the power series about infinity rather than the origin. Consequently we study $f(z)$, where

$$f(z) = F(x), \qquad x = z^{-1}, \tag{1.2.5}$$

so that f(z) has the expansion

$$f(z) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots, \qquad z \to \infty. \tag{1.2.6}$$

Also we, in this thesis, investigate only the case when two polynomials have the same degree, and this leads to

Definition 1.2 *The diagonal Padé approximant of degree* n *to* f *is constructed from Padé polynomials* p_1, p_2 *of degree* $\le$ n *that satisfy*

$$p_1(z) + p_2(z)f(z) = O(z^{-n-1}), \qquad z \to \infty. \tag{1.2.7}$$

The Padé approximant [n/n] *is given by*

$$[n/n] = - p_1(z) \, / \, p_2(z). \tag{1.2.8}$$

Again note that [n/n] is unique although the Padé polynomials may not be. The Definition 1.2 will be used throughout the thesis.

1.3 Applications

It is now well known that Padé approximants have wide applications in numerical analysis, computation, fluid mechanics and many fields in physics, such as statistical physics, scattering physics and other parts of theoretical physics.

A typical example is the application of Padé approximants to critical phenomena. It has been shown that, with the Padé method, it is possible to obtain significant results where previous methods have failed. For a fuller discussion, there are two good reviews of this area by Hunter and Baker [14], and Gaunt and Guttmann [8], and also a recent review article by Guttmann [12].

There has been a number of problems in fluid mechanics for which Padé methods have worked extremely well. These problems have been reviewed by Van Dyke and Guttmann [46].

The applications of Padé approximants in numerical analysis. scattering theory and theoretical physics have been discussed by Baker and Graves-Morris [1], [3] and a number of physicists and applied mathematicians in [34] and [35].

Most of the above applications, although often apparently successful, have been empirical. No mathematical theorems were invoked to prove the convergence of the sequence of calculated approximations, and no rigorous bounds were used to limit the size of the error in any estimate. The aim of this thesis is to contribute to the development of mathematical results that can form a sound basis for practical applications of Padé methods.

1.4 Relation of Pade Approximants to Orthogonal Polynomials

There is an important connection between Padé polynomials and orthogonal polynomials. In fact, for some functions, Padé polynomials are orthogonal polynomials corresponding to an appropriate weight. Details of the relation between Padé and orthogonal polynomials may be found in references [45], [27]. Here we give a brief summary of the important points.

If $f(z)$ is analytic in a neighbourhood of infinity, say in $|z| \geq R$, then from (1.2.7) it is easy to deduce that

$$\int_{\Gamma} f(z)\, p_2(z)\, z^k\, dz = 0, \qquad k = 0,\dots,n-1, \tag{1.4.1}$$

where $\Gamma = \{z\in\mathbf{C}: |z| = R'\}$, $R' > R$. If further it is supposed that $f(z)$ is analytic throughout $\mathbf{C}$ except for an interval $L=\{z\in\mathbf{C}: z \text{ real}, -1\leq z\leq 1\}$ and that the limit of $f(z)$ as $z \to L$ (either side) exists and is integrable, then (1.4.1) leads to

$$\int_{L} \omega(z)\, p_2(z)\, z^k\, dz = 0, \qquad k = 0,\dots,n-1, \tag{1.4.2}$$

where

$$\omega(z) = f_+(z) - f_-(z). \tag{1.4.3}$$

The symbols +, - refer to the left, right sides of L. In general, to define the +, - sides of an arc, we choose a direction for the arc and then the + side will be to the left as we move along the arc in the positive direction.

If $\omega(z)$ happens to be real and non-negative, then (1.4.2) implies that $p_2(z)$ is an orthogonal polynomial of degree n in z corresponding to weight $\omega(z)$.

In terms of $p_2(z)$ it is possible to write $p_1(z)$ as [27]

$$(1.4.4) \qquad p_1(z) = -f(\infty)p_2(z) - (2\pi i)^{-1}\int_\Gamma f(t)(p_2(t) - p_2(z))(t-z)^{-1}dt.$$

With the further assumption above about f(z), this may be transformed to

$$(1.4.5) \qquad p_1(z) = -f(\infty)p_2(z) - (2\pi i)^{-1}\int_L \omega(t)(p_2(t) - p_2(z))(t-z)^{-1}dt.$$

As will be seen in Section 2.2, the classical results of Bernstein and Szegö about the behaviour of orthogonal polynomials of high degree [27] imply corresponding results about the asymptotics of Padé polynomials, at least for certain functions. It turns out that the asymptotic formulas so obtained, if written in a suitable form, may be generalized to cases for which the method of Szegö is not applicable.

1.5 Convergence of Pade Approximants and Asymptotics of Pade Polynomials

The classical convergence results on diagonal Padé approximants are now very old and many of them were obtained within the framework of the analytic theory of continued fractions. Those classical theorems were concerned with important but very restricted classes of analytic functions [11]. Except for some special cases, little was known until a few years ago about the convergence of the sequence of diagonal Padé approximants, or about the closely related question of the asymptotic behaviour of Padé polynomials. Now, however, the outline of a theory is visible, although many details remain to be filled in.

The most important results on convergence were obtained in the last twenty years. The first one was Nuttall's theorem (1970) on convergence in measure of the diagonal Padé approximants to functions meromorphic in the complex plane [21]. A sequence of approximants $f_n(z)$ converges in measure to $f(z)$ in the set $\sum$ if the measure of the set within $\sum$ on which $|f_n(z)-f(z)|$ is greater than any given prescribed (small) number approaches zero as $n \to \infty$. Later, Pommerenke (1973) obtained [36] a more general result on convergence in capacity for single-valued analytic functions with singularity sets of capacity zero. The definition of convergence in capacity is as above with measure replaced by capacity [13]. Along these lines, there were also Nuttall's results [23], [24] for classes of multi-valued analytic functions with a finite set of branch points and also some related results from Stahl [40], [42], Gonchar [9], [10], Lubinsky [18], [19], Saff [19], Wallin [47], and others.

In this thesis, we are particularly interested in investigating the convergence theory for diagonal Padé approximants to classes of functions with branch points. The most important idea in this convergence theory is that diagonal Padé approximants "choose" a certain preferred location of cuts S with which to make the function

$f(z)$ single-valued. Away from S, the sequence of diagonal Padé approximants converges to $f(z)$. However, the convergence is not in general uniform in z, but is of a weaker type – convergence in capacity seems to be natural. What happens is that almost all the zeros and poles of $[n/n]$ approach S as $n \to \infty$, but that sometimes almost coincident pairs of zeros and poles are present away from S. The location of these pairs depends on n. The existence of these pairs, sometimes known as spurious poles and zeros, complicates convergence proofs.

Most proofs of the ideas outlined above have been based on a study of the asymptotics of Padé polynomials as the degree $n \to \infty$, from which the convergence of Padé approximants is easily deduced. There are several different methods of obtaining the asymptotics of Padé polynomials, each with their own assumptions about $f(z)$. We will summarize some previous work on the asymptotics of Padé polynomials in Chapters 2 and 3.

No method has so far been able to prove results as general as those which are conjectured by Nuttall [27] to hold. This thesis goes some way in the development of more powerful methods of proof.

CHAPTER 2: PREVIOUS WORK ON CONVERGENCE AND ASYMPTOTICS

A completely general theory of the asymptotics of Padé polynomials does not yet exist. However, there are some functions for which the asymptotics can be worked out more or less explicitly, and others where special techniques allow the derivation of useful results. Some of these cases will be discussed in this chapter. These cases will introduce ideas that will be needed in the explanation of our new, more general methods which are described in Chapters 4-8.

2.1 Asymptotics of Orthogonal Polynomials

We return to the case discussed in Section 1.4, where $f(z)$ is such that (1.4.2) holds and $\omega(z)$ is real and non-negative on L. In this case, all the n zeros of $p_2(z)$ are distinct and lie on L. Suppose that the branch points of $f(z)$ at $z = -1, 1$ are of dominantly square root type and that

$$\omega(z) = (1-z^2)^{-1/2}\sigma(z), \tag{2.1.1}$$

with $\sigma(z) \geq 0$, and that the integrals

$$\int_0^\pi \sigma(\cos\theta)\,d\theta, \qquad \int_0^\pi \left|\log\sigma(\cos\theta)\right|d\theta \tag{2.1.2}$$

exist. Then a theorem of Szegö [45] shows that $p_2(z)$ is unique up to normalization, which may be chosen so that

$$(2.1.3) \qquad p_j(z) \underset{n\to\infty}{\sim} \chi_j(z), \qquad z \notin L, \quad j = 1, 2.$$

The functions $\chi_j(z)$ are analytic in $\mathbb{C}\setminus L$, and are proportional to z^n at ∞. Explicitly, with a suitable choice of normalization, $\chi_2(z)$ is given by

$$(2.1.4) \qquad \chi_2(z) = h(z)^{-1},$$

where the function $h(z)$, analytic in $\mathbb{C}\setminus L$, may be written as

$$(2.1.5) \quad h(z)=\exp\left[(2\pi i)^{-1}X(z)^{1/2}\int_L X_+(t)^{-1/2}\log\sigma(t)(t-z)^{-1}dt\right]\exp\left[n\phi(z)\right].$$

Here

$$(2.1.6) \qquad X(z) = z^2-1,$$

and $X_+(t)^{-1/2}$ denotes the value of $X(z)^{-1/2}$ on the left hand side of L. The function $\phi(z)$ is given by

$$(2.1.7) \qquad \phi(z) = \log\left[z - (z^2-1)^{1/2}\right],$$

with the branch being chosen so that $\phi(z) \sim -\log z$ near $z = \infty$. We note that $\mathrm{Re}\phi(z) < 0$ for $z \notin L$. The function $\chi_1(z)$, is given by

$$(2.1.8) \qquad \chi_1(z) = - f(z)\chi_2(z), \qquad z \in \mathbb{C}\setminus L.$$

With more restrictions on $\sigma(z)$, the asymptotic behaviour of the polynomials on L may be given [27]. Thus, if $\sigma(z)$ is strictly positive, $z \in L$, and satisfies the condition

$$(2.1.9) \qquad \left|\sigma(\cos(\theta+\delta)-\sigma(\cos\theta)\right| < \text{const.}\left|\log\delta\right|^{-1-\lambda}, \qquad \lambda > 1,$$

then

$$(2.1.10) \qquad p_j(z) = \chi_{j+}(z) + \chi_{j-}(z) + O\left[(\log n)^{-\lambda}\right], \qquad z \in L, \quad j = 1,2.$$

There is a general conjecture about the asymptotics of Padé polynomials [27]. The conjecture holds in this case, as may be checked by defining

$$(2.1.11) \qquad R_0(z) = X(z)^{-1/2}\,\chi_2(z)^{-1}, \qquad z \notin L.$$

It will be found that

$$(2.1.12) \qquad \chi_1(z) + f(z)\chi_2(z) = 0, \qquad z \notin L$$

$$(2.1.13) \qquad \begin{aligned} \chi_{1+}(z) + f_-(z)\chi_{2+}(z) &= R_{0-}(z) \\ \chi_{1-}(z) + f_+(z)\chi_{2-}(z) &= R_{0+}(z) \end{aligned} \qquad z \in L$$

which according to the conjecture are the equations to be solved to find $\chi_j(z)$, which give the asymptotic behaviour of $p_j(z)$.

Because $p_2(z)$, $\chi_2(z)$ have no zeros in $\mathbb{C}\backslash L$ it is easy to show in this case, using (1.4.5), that the diagonal Padé approximants converge to f(z) uniformly in any closed set not intersecting L.

2.2 Complex Weight on L

It was realized first by Baxter [4] and later, using a different method, by Nuttall [22], that the asymptotics of the polynomials defined by (1.4.2) are much the same even if the weight $\omega(z)$ is complex. The essential requirement is that it be non-zero. Nuttall [22] showed that for $\sigma(z)$ with $|\sigma(z)| > A > 0$, $z \in L$, and satisfying (2.1.9), the asymptotic formulas (2.1.3), (2.1.10) still hold. In this case the zeros of $p_2(z)$ may not all lie on L, but will approach L as $n \to \infty$. For large enough n the polynomials are unique. The convergence of the Padé approximants is as above.

2.3 A Special Case Functions Meromorphic on a Riemann Surface with Two Sheets

We consider functions meromorphic on a Riemann surface with two sheets. Although we can handle any such functions, we restrict our attention to one case, a function closely related to an important historical example investigated by Dumas [6], and recently treated by Nuttall [29].

Suppose that

(2.3.1)
$$X(z) = \prod_{j=1}^{4} (z-b_j)$$

with $\{b_j\}$ being complex, distinct points. The Riemann surface $\mathbf{R}$ corresponding to the equation

(2.3.2)
$$y^2 = X(z)$$

consists of two copies of the extended complex z-plane with arbitrary, non-intersecting cuts joining b_1, b_2 and b_3, b_4, say. On the surface $\mathbf{R}$, the function y is single-valued, and we assume that near ∞ on sheet 1, ie. $\infty^{(1)}$, $y \sim z^2$. We define f(z) by

(2.3.3)
$$f(z) = y - (z^2 - (z/2) \sum_{j=1}^{4} b_j)$$

so that near $\infty^{(1)}$, $f(z) \sim$ const. The Padé polynomials to f(z), expanded about $\infty^{(1)}$, are defined by (see (1.2.7))

(2.3.4)
$$p_1(z) + p_2(z)f(z) = O(z^{-n-1}), \qquad z \to \infty^{(1)}.$$

and we use the remainder function R(z) given by

(2.3.5)
$$R(z) = p_1(z) + p_2(z)f(z).$$

On $\mathbf{R}$ there is at every point an appropriate local variable. Near ∞, we use z^{-1} and near $z = b_j$ we use $(z-b_j)^{1/2}$. Otherwise the variable z will do. A meromorphic function on $\mathbf{R}$ is a single-valued function on $\mathbf{R}$ which is everywhere analytic in the local variable except at poles. The meromorphic functions on $\mathbf{R}$ are the functions which are rational in y, z. They have as many poles as zeros, taking account of multiplicities.

The surface $\mathbf{R}$ that we are considering in this example is topologically equivalent to a torus, with genus 1. On such a surface there is one relation between the poles and zeros of a meromorphic function. This relation is given in terms of the Abelian differential of the first kind, $w(z)dz$, where

$$w(z) = y^{-1} \tag{2.3.6}$$

Apart from a constant factor, $w(z)$ is the only meromorphic function for which $\int w(z)dz$ exists for any path on $\mathbf{R}$.

Suppose that $\bar{C}_1$, $\bar{C}_2$ are two independent closed curves on $\mathbf{R}$, for example a loop round b_1, b_2 and a loop round b_1, b_3 in the z-plane. Then the periods of w are

$$\bar{\Omega}_j = \int_{\bar{C}_j} w(z)dz, \qquad j = 1,2. \tag{2.3.7}$$

The integral of w round any closed path on $\mathbf{R}$ may be written as a linear combination of $\bar{\Omega}_1$, $\bar{\Omega}_2$ with integer coefficients.

Now if $a_1,\dots a_k$ and $c_1,\dots c_k$ are points on $\mathbf{R}$ which are the poles, zeros respectively of a meromorphic function, Abel's theorem states that

$$\sum_{j=1}^{k} \int_{a_j}^{c_j} w(z)dz = n_1\bar{\Omega}_1 + n_2\bar{\Omega}_2 \tag{2.3.8}$$

for some integers n_1, n_2. (See [39] for more information about Riemman surfaces and related questions.)

With this information we can proceed to the construction of the Padé polynomials, which we do by first finding $R(z)$. The meromorphic function $R(z)$ has a pole only at $\infty^{(2)}$, and this pole is of order $n + 2$, since $f(z)$ has a pole of order 2 at $\infty^{(2)}$. Also $R(z)$ has a zero of order $n + 1$ at $\infty^{(1)}$ on account of (2.3.4). Thus there is one zero remaining, say $c_n \in \mathbf{R}$. This point satisfies

$$\int_{\infty^{(2)}}^{c_n} w(z)dz = -(n + 1) \int_{\infty^{(2)}}^{\infty^{(1)}} w(z)dz + n_1\bar{\Omega}_1 + n_2\bar{\Omega}_2 \tag{2.3.9}$$

This equation may be solved uniquely by inverting the elliptic integral on the left. That is we use the function $Z(u)$ defined by

$$\int_{\infty^{(2)}}^{Z(u)} w(z)dz = u \tag{2.3.10}$$

The function $Z(u)$ is a elliptic function, doubly periodic with periods $\bar{\Omega}_1$, $\bar{\Omega}_2$, and an explicit expression for c_n can be found.

Having found c_n we can construct $R(z)$. This may be done as in Dumas [6] with Weierstrass σ functions, or as solved by Nuttall [27] by using Abelian integrals of the third kind.

Given two distinct points z_1, $z_2 \in \mathbb{R}$, there exists a unique Abelian differential $dE(z_1, z_2)$ of the third kind whose only singularities are simple poles at z_1, z_2, with residues 1, -1, respectively, and such that the periods of the integral

$$(2.3.11) \qquad E(z_1, z_2; z) = \int^z dE(z_1, z_2)$$

are pure imaginary [39]. It follows that

$$(2..3.12) \qquad R(z) = \text{const.}\exp\left[(n+1)E(\infty^{(1)}, \infty^{(2)}; z) + E(c_n, \infty^{(2)}; z)\right].$$

For the case in question

$$(2.3.13) \qquad dE(\infty^{(1)}, \infty^{(2)}; z) \equiv dE(z) = X(z)^{-1/2}(z-a)dz.$$

We set

$$(2.3.14) \qquad \phi(z) = \int_{b_1}^z dE = \int_{b_1}^z X(t)^{-1/2}(t - a)dt$$

and the condition of imaginary periods is equivalent to

$$(2.3.15) \qquad \mathrm{Re}\,\phi(b_2) = \mathrm{Re}\,\phi(b_3) = 0$$

from which a may be determined uniquely.

If we take

$$S = \{z \in \mathbb{C}: \operatorname{Re}\phi(z) = 0\} \tag{2.3.16}$$

to be the cuts separating the two sheets of $\mathbb{R}$, then we define the sheets by

$$\begin{aligned} \operatorname{Re}\phi(z) &< 0, \qquad z \in \text{sheet } 1. \\ \operatorname{Re}\phi(z) &> 0, \qquad z \in \text{sheet } 2. \end{aligned} \tag{2.3.17}$$

To express $p_2(z)$ in terms of $R(z)$, we use (2.3.5) to give

$$\begin{aligned} p_1(z) + p_2(z) f(z^{(1)}) &= R(z^{(1)}) \\ p_1(z) + p_2(z) f(z^{(2)}) &= R(z^{(2)}) \end{aligned} \tag{2.3.18}$$

and to solve for $p_2(z)$, i.e.

$$p_2(z) = \frac{R(z^{(1)}) - R(z^{(2)})}{f(z^{(1)}) - f(z^{(2)})} . \tag{2.3.19}$$

Now, for large n, $z \notin S$, it may be shown that $R(z^{(2)}) \gg R(z^{(1)})$, unless c_n is on sheet 2 near to z. With this exception, it follows that

$$p_2(z) \underset{n\to\infty}{\sim} \text{[illegible]}(z) = -R(z^{(2)})\left[f(z^{(1)}) - f(z^{(2)})\right]^{-1}, \qquad z \notin S. \tag{2.3.20}$$

The behaviour of $p_1(z)$ may be obtained in a similar manner.

In this case also, the equations of the Nuttall's conjecture [27] are satisfied. The set S takes the place of L in Section 2.1. The set consists of several analytic arcs, generally two in number. We define

$$R_0(z) = R(z), \quad z \in \text{sheet } 1. \tag{2.3.21}$$

It may be shown that

$$\chi_1(z) + f(z)\chi_2(z) = 0 \qquad z \notin S \tag{2.3.22}$$

$$\left.\begin{aligned} \chi_{1+}(z) + f_-(z)\chi_{2+}(z) &= R_{0-}(z) \\ \chi_{1-}(z) + f_+(z)\chi_{2-}(z) &= R_{0+}(z) \end{aligned}\right\} \quad z \in S \tag{2.3.23}$$

which are the equations of the conjecture in this case.

It is also possible to demonstrate that

$$p_2(z) \underset{n\to\infty}{\sim} \chi_{2+}(z) + \chi_{2-}(z), \qquad z \in S \tag{2.3.24}$$

The above results show that, apart from one possible exception, the zeros of $p_2(z)$ (and $p_1(z)$) approach S as $n \to \infty$. If c_n is on sheet 2, $p_j(z)$ will have a zero near this point for large n. The polynomials will certainly be unique unless c_n is near $\infty^{(2)}$. The Padé approximant will be a good approximation to $f(z)$ for large n, $z \notin S$, unless z is near c_n when this point is on sheet 2.

2.4 An Example of Laguerre

So far, proofs of the more detailed form of the asymptotic behaviour of Padé polynomials have applied in general only to functions with branch points of square root type, such as the examples of Sections 2.1 - 2.3. It appears likely that most of the asymptotic results carry over to other types of branch points, so that it is worthwhile to investigate examples of such functions. An interesting example was first looked at by Laguerre [16] and cleared up recently by Nuttall [29]. It provides helpful hints about results to be expected in general.

We consider the function

$$f(z) = \prod_{j=1}^{3} (z - b_j)^{\nu_j} \tag{2.4.1}$$

where b_j, $j = 1, 2, 3$, are distinct, finite, noncollinear points in the complex plane, and the complex numbers ν_j, not integers, satisfy

$$\sum_{j=1}^{3} \nu_j = 0 \tag{2.4.2}$$

As usual our main concern is to determine the asymptotic behavior of $p_1(z)$, $p_2(z)$ as $n \to \infty$, from which the convergence of the Padé approximants is easily deduced. The basic tool for this is a second order differential equation with polynomial coefficients and such an equation was first obtained by Laguerre [16]. He showed that $p_2(z)$ satisfies this differential equation and obtained nonlinear recurrence relations relating the coefficients in this equation, but he was unable to solve the recurrence relations or to obtain the asymptotic behaviour of the coefficients.

Instead of considering $p_2(z)$, Nuttall has obtained a similar differential equation satisfied by the remainder function $R(z)$. The derivation is based on the theory of Riemann modules [37], which was recently described by Chudnovsky [5].

It turns out that $R(z)$ satisfies the differential equation

$$\theta\pi_1 R'' + \pi_3 R' + \pi_2 R = 0 \tag{2.4.3}$$

where primes indicate derivatives,

$$\theta(z) = \prod_{j=1}^{3} (z - b_j) \tag{2.4.4}$$

and $\pi_j(z)$, $j = 1, 2, 3$, are polynomials of degree corresponding to their subscripts.

With the help of the Liouville-Green approximation to the solution of (2.4.3) [31] and the results of Stahl on weak asymptotics [42], it is possible to work out approximate values for the coefficients in the polynomials $\pi_j(z)$ as $n \to \infty$. The results are again expressed in terms of a set of arcs S which is determined by using the function

$$\phi(z) = \int_{b_1}^{z} \theta(t)^{-1/2}(t - a)^{1/2}dt \tag{2.4.5}$$

with the point a found by solving

$$(2.4.6) \qquad \mathrm{Re}\phi(b_2) = \mathrm{Re}\phi(b_3) = 0.$$

Again we define

$$(2.4.7) \qquad S = \{z \in \mathbf{C}: \mathrm{Re}\phi(z)=0\}.$$

It is found that S is a set of three analytic arcs joining the points b_j, j = 1, 2, 3, to the point a. The arcs meet at angles of $2\pi/3$. This set S may be characterized as the unique connected set of minimum capacity [28], [25] containing the points b_1, b_2, and b_3.

It was shown [29] that polynomial asymptotics are given by functions $\chi_j(z)$, $z \notin S$, j = 1, 2, analytic outside S, which satisfy

$$(2.4.8) \qquad \chi_1(z) + f(z)\chi_2(z) = 0 \qquad z \notin S$$

$$(2.4.9) \qquad \begin{aligned} \chi_{1+}(z) + f_-(z)\chi_{2+}(z) &= R_{0-}(z) \\ \chi_{1-}(z) + f_+(z)\chi_{2-}(z) &= R_{0+}(z) \end{aligned} \qquad z \in S$$

Here, $R_0(z)$ is analytic outside S. At ∞ we have

$$(2.4.10) \qquad \chi_j(z) = O(z^n), \qquad j = 1, 2,$$

$$(2.4.11) \qquad R_0(z) = O(z^{-n-1}).$$

These equations may be solved as in [29]. It turns out that there is a point c_n, varying with n, at which either $\chi_1(c_n) = \chi_2(c_n) = 0$ or $R_0(c_n)=0$. A formula for c_n can be found in terms of elliptic functions.

The results are that

$$(2.4.15) \qquad p_j(z) \underset{n\to\infty}{\sim} \chi_j(z), \qquad z \notin S, \qquad j = 1, 2$$

provided that z is not near c_n if $\chi_j(c_n)=0$. On S, but not near b_1, b_2, b_3 or a, we have

$$(2.4.16) \qquad p_j(z) \underset{n\to\infty}{\sim} \chi_{j+}(z) + \chi_{j-}(z), \qquad z \in S.$$

In this case it may be proved that the polynomials are unique for all n. We remark that the predictions of Nuttall's conjecture are satisfied in this case.

To find the form of $p_j(z)$ near the four points excluded above, we may go back to the differential equation (2.4.3). We consider only the general case when c_n is not near ∞ or the points b_1, b_2, b_3, and a. The work of [29] shows that for large n, R(z) is given by

$$(2.4.17) \qquad R(z) = \xi(z)^{1/2}u(z)$$

where

$$(2.4.18) \qquad \xi(z) = f(z)(z - c_n)\theta(z)^{-1}$$

The function u(z) satisfies

$$u''(z) = \lambda(z)u(z) \tag{2.4.19}$$

where

$$\lambda(z) = (n + 1/2)^2(z - a)\theta(z)^{-1} + O(1). \tag{2.4.20}$$

The behaviour of R(z) near $z = b_j$ or $z = a$ is found by solving (2.4.19) approximately near the appropriate point. A rigorous discussion may be found in Olver [31]. Thus near $z = b_1$ we write

$$\lambda(z)\approx(n+1/2)^2(b_1-a)(b_1-b_2)^{-1}(b_1-b_3)^{-1}(z-b_1)^{-1}+((\nu_1^2-1)/4)(z-b_1)^{-2} \tag{2.4.21}$$

The required solution is

$$R(z) = \text{const.}(z-b_1)^{\nu_1/2}\exp(n\alpha(z-b_1)^{1/2})K_{\nu_1}(n\alpha(z-b_1)^{1/2}) \tag{2.4.22}$$

where α is a known constant. The constant in (2.4.22) may be determined by matching the form of K_{ν_1} valid for large argument to the form of the approximation $R_0(z)$ valid near $z=b_1$. This is permissible since we make the argument of K_{ν_1} large by taking n large while keeping $z-b_1$ small.

Similarly, near z=a, we use

$$\lambda(z) \approx (n + 1/2)^2\theta(a)^{-1}(z - a) \tag{2.4.23}$$

giving

$$R(z) = \text{const.}(z-a)^{1/2}\exp(n\beta(z-a)^{3/2})K_{1/3}(n\beta(z-a)^{3/2}), \tag{2.4.24}$$

with β a known constant. The constant may be determined as above. Note there will be different values for the constant in the solutions in the three sectors near $z = a$ bounded by the arcs of S.

CHAPTER 3: FUNCTIONS WITH BRANCH POINTS--WEAK AND STRONG ASYMPTOTICS

In each of the examples described in Chapter 2, a set of arcs S appears in the asymptotic behaviour of the diagonal Padé polynomials. For a large class of functions with branch points, there is a unique such set [25], [41] and this is probably true for any analytic function with branch points that does not have a natural boundary. We review the nature of the possible sets S and survey previous results on the asymptotics of diagonal Padé polynomials for functions with branch points.

3.1 Nature of Preferred Sets

The possible sets S that can arise in studying the asymptotics of Padé polynomials for functions with branch points are members of a class **M**. There is a number of possible ways of characterizing these sets [27], [41]. We choose an analytic approach based on Abelian integrals. Associated with every S is a two-sheeted Riemann surface **R** that can be described by the equation

$$y^2 = X(z) \tag{3.1.1}$$

$$= \prod_{j=1}^{2m} (z - b_j)$$

where $b_1,\ldots,b_{2m}$ are distinct finite points in the complex plane. Given a monic polynomial $Y(z)$ of degree m-1, the function $\phi(z)$ is defined by

$$\phi(z) = \int_{b_1}^{z} Y(t)X(t)^{-1/2}dt. \tag{3.1.2}$$

The polynomial Y is defined uniquely by the requirement that the periods of $\phi(z)$ are pure imaginary, which is equivalent to

$$\mathrm{Re}\left\{\int_{b_1}^{b_{j+1}} Y(t)X(t)^{-1/2}dt\right\} = 0, \qquad j=1,\ldots,2(m-1). \tag{3.1.3}$$

The paths in (3.1.3) may be chosen in any way, provided that $X(t)^{-1/2}$ is continuous along each path. The set S associated with the Riemann surface $\mathbb{R}$ (3.1.1) is defined by

$$S = \left\{z \in \mathbb{C}:\ \mathrm{Re}\phi(z) = 0\right\}. \tag{3.1.4}$$

In (3.1.4) it does not matter which evaluation of $\phi(z)$ is used on account of (3.1.3).

It may be shown [24] that any set S consists of a number of analytic arcs. If a zero of $Y(z)$ of multiplicity q coincides with b_k, then 2q+1 arcs end at b_k, but if the zero belongs to S and is not coincident with any b_j, $j = 1,\ldots,2m$, then q + 1 arcs intersect at the zero. The set S consists of one or more components. In the complement of S, which is connected, $\mathrm{Re}\phi(z)$ is single-valued.

In this thesis, we shall be concerned with just three examples of sets S, but the techniques of analysis appear to be extendable to the general case. The three sets are those encountered in Sections 2.2, 2.3, 2.4. We shall call them S_1, S_2, S_3. For these sets we have

$$(3.1.5) \qquad S_1: \quad X(z) = z^2 - 1, \qquad \phi(z) = \int_1^z X(t)^{-1/2}dt.$$

$$(3.1.6) \qquad S_2: \quad X(z) = \prod_{j=1}^{4} (z-b_j), \qquad \phi(z) = \int_{b_1}^{z} (t-a)X(t)^{-1/2}dt.$$

$$(3.1.7) \qquad S_3: \quad X(z) = (z-a)\prod_{j=1}^{3} (z-b_j), \qquad \phi(z) = \int_{b_1}^{z} (t-a)X(t)^{-1/2}dt.$$

Thus $S_1 = L$, the line segment joining $z = -1, 1$. The set S_2 consists of two disjoint analytic arcs joining pairs b_1, b_2 and b_3, b_4 (by choice of notation), and the point a is not on S_2. The set S_3 consists of three analytic arcs joining a to b_1, b_2, b_3.

3.2 Weak Asymptotics

The study of weak asymptotics involves the behaviour of $|p_j(z)|^{1/n}$, $j = 1, 2$, as the degree n of the polynomials approaches ∞. The results are enough to demonstrate the convergence in capacity of the corresponding Padé approximants. The principal work in this area has been done by Stahl [42], [43], [44]. He assumes that the function $f(z)$, with certain branch points and essential singularities is analytic near $\infty^{(1)}$ (the point at ∞ on the sheet on which we expand $f(z)$ to obtain

$p_j(z)$). All its singularities are contained in a compact set $E \subset \mathbf{C}$ of capacity zero, and the function $f(z)$ may be continued analytically along any path in $\hat{\mathbf{C}} \backslash E$ beginning at $\infty^{(1)}$. There is a unique set $S \subset \mathbf{M}$ which satisfies the following conditions.

1) The continuation of $f(z)$ from $\infty^{(1)}$ is single-valued in $\mathbf{C} \backslash S$.
2) The discontinuity of $f(z)$ across S is non-zero except possibly for a set of capacity zero.

For functions $f(z)$ satisfying the above conditions, Stahl [42] has proved the following

Theorem 3.1 *For each* n *a choice of normalization may be found such that, if* $V \subset \mathbf{C}$ *is a compact set not intersecting* S, *for every* $\varepsilon > 0$,

$$(3.2.1) \qquad \lim_{n\to\infty} \mathrm{Cap}\left\{z\in V:\ |n^{-1}\log|p_j(z)| + \mathrm{Re}\phi(z)|>\varepsilon\right\} = 0, \qquad j=1,\ 2,$$

where Cap means the logarithmic capacity of the set.

From this theorem, it may be deduced that diagonal Padé approximants converge in capacity to $f(z)$ in any compact set not intersecting S. It is also possible to obtain an analogous result about the remainder function $R(z)$ [29].

3.3 Strong Asymptotics

The important role of the preferred set S in the convergence of diagonal Padé approximants was first discovered by Nuttall [24] in

1973. The discovery was made by analyzing the asymptotics of the Padé polynomials for various examples, such as those described in Chapter 2, and a compact universal way of characterizing the form of these polynomials for large n was suggested. This is known as Nuttall's conjecture on the strong asymptotics of Padé polynomials. The essentials of the conjecture may be given as follows.

Nuttall's Conjecture on Strong Pade Asymptotics

Suppose that we are given a preferred set $S \in \mathcal{M}$ and a function

$$(3.3.1) \qquad f(z) = f_0 + (2\pi i)^{-1}\int_S \omega(t)(t-z)^{-1}dt, \qquad f_0 \text{ is a const.}$$

Let $\chi_1(z)$, $\chi_2(z)$, $R_0(z)$ be functions analytic in z for $z \in \mathbb{C}\backslash S$, with

$$(3.3.2) \qquad \begin{aligned} \chi_j(z) &\sim C_j z^n \qquad z \to \infty, \quad j=1,\ 2, \\ R_0(z) &\sim C_0 z^{-n+1} \qquad z \to \infty. \end{aligned}$$

satisfying

$$(3.3.3) \qquad \chi_1(z) + f(z)\chi_2(z) = 0 \qquad z \notin S,$$

$$(3.3.4) \qquad \begin{aligned} \chi_{1+}(z) + f_-(z)\chi_{2+}(z) &= R_{0-}(z) \\ \chi_{1-}(z) + f_+(z)\chi_{2-}(z) &= R_{0+}(z) \end{aligned} \qquad z \in S.$$

Then, subject to certain conditions on $\omega(z)$, $z \in S$, and on the behaviour of $\chi_j(z)$, j=1, 2, and $R_0(z)$, $z \in S$, we have

C1) the functions $\chi_j(z)$, j=1, 2, $R_0(z)$ are unique up to a common constant factor.

C2) except near zeros of $\chi_j(z)$,

(3.3.5) $$p_j(z) \underset{n\to\infty}{\sim} \chi_j(z), \qquad j=1,\ 2, \quad z \notin S,$$

C3) except for the vicinity of certain exceptional points on S,

(3.3.6) $$p_j(z) \underset{n\to\infty}{\sim} \chi_{j+}(z) + \chi_{j-}(z), \qquad j=1,\ 2, \quad z \in S.$$

We have indicated in Chapter 2 how the examples discussed there obey the conjecture. The conjecture is also consistent with Stahl's work on weak asymptotics.

If the conjecture is to make sense, there can be no more than one set $S \in \mathbf{M}$ corresponding to a given function f(z). A proof of the conjecture would lead to this conclusion, because a Padé polynomial cannot have two different asymptotic forms. Alternatively, a direct argument about the uniqueness of S has been given by Nuttall [27].

The outstanding problems associated with the conjecture are

P1) determine the weakest possible conditions on $\omega(z)$, $z \in S$.

P2) determine the appropriate restrictions on the behaviour of $\chi_j(z)$, $R_0(z)$.

P3) prove the correctness of (3.3.5), (3.3.6)

P4) determine the behaviour of $p_j(z)$ near the exceptional points.

We outline what is the current thinking about these problems.

T1) Proofs are easiest when $\omega(z)$, $z \in S$, is not zero, is adequately smooth, and behaves like $(z-b)^{-1/2}$ at a point b where a single arc of S ends. None of these restrictions is probably necessary.

T2) With the above assumption on $\omega(z)$, it is required that $\chi_1(z)$, $\chi_2(z)$, $(z-b)^{1/2}R_0(z)$ be bounded at each end point b.

T3) Apart from rather special cases, a proof exists only when S consists of a number of non-intersecting arcs and the conditions of T1) above are imposed.

T4) Exceptional points include these points b where a single arc of S ends and $\omega(z)$ does not behave like $(z-b)^{-1/2}$, and also points where several arcs of S intersect.

The proof referred to in T3) above by Nuttall and Singh [24] used an extension of the Bernstein-Szegö integral equation. In this method, the weight $\omega(z)$ is approximated by $X_+(z)^{-1/2}$ multiplied by the reciprocal of a polynomial in z. Padé polynomials for this weight may be constructed explicitly. The integral equation giving the required polynomial has a kernel that involves the difference between the approximate weight and $\omega(z)$. With the appropriate assumptions, mentioned in T1) above, this kernel is small, and the result follows.

In Chapter 4, we describe the progress made in this thesis using a new method of solving some of the problems listed above.

CHAPTER 4: SUMMARY OF RESULTS

The long term goal in the subject under consideration is the proof of Nuttall's conjecture (assuming it to be correct) in as general a case as possible. Recently, Nuttall [30] proposed a new technique for the asymptotic analysis of Padé polynomials. The method involves a singular integral equation unrelated to the Bernstein-Szegö type integral equation used previously by Nuttall and Singh [24]. It introduces in a natural way the functions $\chi_j(z)$ used in the conjecture of Section 3.3.

Nuttall [30] derived and applied the new integral equation in the case of Section 2.2, where the set S is the line segment L={z real, $-1\le z\le 1$}, and the weight $\omega(z)=(z^2-1)^{-1/2}\sigma(z)$, with $\sigma(z)$ non-vanishing on L. The aim of this thesis is to show how to extend the derivation and application of the new method to other sets $S\in\mathbf{M}$ and to weight functions $\omega(z)$ that do not behave like $(z-b)^{-1/2}$ near an arc-end point b of S. In the subsequent four chapters, we have obtained the following results.

Chapter 5. - The new integral equation was previously [30] written in the language of the Riemann surface associated with L, the particular set $S \in \mathbf{M}$ being studied. For more complicated sets S, this surface has genus > 0 and complications arise. We have derived a new form of the integral equation which involves an integral over $S \in \mathbf{C}$. The unknown quantity U(z) is related to the remainder function R(z) (see (5.1.5), (5.1.8)). In (5.1.15), (5.2.5) and (5.2.8) this equation is given for some representative sets S that will be studied in later chapters. We also give a formula which effectively expresses the Padé polynomial $p_2(z)$ in terms of $U(z)$ (see (5.1.16), (5.2.9) and (5.2.10)). This new version of the integral equation allows us to overcome the difficulties referred to above.

Chapter 6. - To analyze the situation in which the weight function $\omega(z)$ does not behave like $(z-b)^{-1/2}$ at a single arc end b, we have studied an example in which the preferred set is L and the dominant behaviour of $\omega(z)$ at the end point $z = 1$ is $(z-1)^{\nu-1/2}$, $0 < \nu < 1/2$. Otherwise, $\omega(z)$ is chosen to make the analysis as simple as possible. Precise assumptions are stated in Section 6.2.

We have proved Nuttall's conjecture for this case, in the process resolving problems P3) and P4) in Chapter 3. We need the definitions of $\chi_1(z)$, $\chi_2(z)$ given in Section 5.1. and $\phi_1(x)$ given in (6.4.6). In part iii) of Theorem 4.1 below, if $\mathrm{Im}z > 0$, $\chi_{jc}(z)$ means the value of $\chi_j(z)$ after analytic continuation upwards through L, and $\phi_{1c}((z-1)^{1/2})$ means the value of $\phi_1((z-1)^{1/2})$ after analytic continuation from $\mathrm{Re}z>1$ clockwise round $z=1$, and similarly for $\mathrm{Im}z<0$.

Theorem 4.1 *When* $f(z)$ *satisfies the assumptions of Section 6.2, the Padé polynomials are unique for large* n *and their normalization may be chosen so that*

i) $$p_j(z) \underset{n\to\infty}{\sim} \chi_j(z), \qquad j=1,\ 2,$$

uniformly for z *in any compact set not intersecting* L.

ii) $$p_j(z) \underset{n\to\infty}{=} \chi_{j+}(z) + \chi_{j-}(z) + o(1), \qquad j=1,\ 2,$$

uniformly for z *in any compact subset of* L *not containing* $z=1$.

iii) $$p_j(z) \underset{n\to\infty}{=} \chi_j(z)\phi_{1c}((z-1)^{1/2}) + \chi_{jc}\phi_1((z-1)^{1/2}) + o(1), \quad j=1,\ 2,$$

for $|z-1|<\varepsilon_1$, ε_1 *suitably small, fixed.*

Note that $z = 1$ is the only exceptional point in this case. Values of ν outside the range given could be treated after modificaitons to the method.

Chapter 7. - This chapter rederives results that are effectively contained in the work of Nuttall and Singh [24]. The case studied is that when the preferred set has the form S_2 (see (3.1.6)), which consists of two disjoint arcs. The main complication of this case is the fact that, in general, either $R_0(z)$ or $\chi_2(z)$ (and $\chi_1(z)$) of the functions needed in Nuttall's conjecture (Section 3.3) has a zero at a point $c_n \notin S$. In the latter alternative, the polynomials $p_1(z)$, $p_2(z)$ will, for large n, have a zero near c_n. Their remaining zeros approach S. This is all as in the example of Section 2.3.

With favourable assumptions on the form of $f(z)$, given in Section 7.1, we are again able to relate the asymptotics ot $p_1(z)$, $p_2(z)$ to functions $\chi_1(z)$, $\chi_2(z)$, defined in Section 7.2. We have

Theorem 4.2 *When* $f(z)$ *satisfies the assumptions of Section* 7.1, *and* n *is such that* c_n, *given in Lemma* 7.1, *is not near infinity, the Padé polynomials are unique for large n, and their normalization may be chosen so that*

i) $$p_j(z) \underset{n\to\infty}{=} \chi_j(z), \qquad j=1,\ 2,$$

uniformly for z in any compact set not intersecting S, *except for z near* c_n *in the case when* $\chi_j(c_n) = 0$.

ii) $$p_j(z) \underset{n\to\infty}{=} \chi_{j+}(z) + \chi_{j-}(z) + o(1), \qquad j=1,\ 2,$$

uniformly for $z \in S$.

In this case there are no exceptional points.

Chapter 8. - The aim of this chapter is to illustrate how to analyze the situation when the preferred set S contains several arcs which meet at a point. The example studied is the set S_3 of (3.1.7) which contains three arcs meeting at equal angles at a point a, an exceptional point for Nuttall's conjecture (Section 3.3). We make favourable assumptions about $f(z)$. The results are expressed in terms of functions $\chi_1(z)$, $\chi_2(z)$ and $R_0(z)$, defined in terms of the function $h(z)$ of Section 8.2 by equations (7.4.1). These functions obey equations (3.3.3),(3.3.4) of the conjecture. There is again a zero c_n with properties exactly as described in Chapter 7. The results may be stated as

Theorem 4.3 *When* $f(z)$ *satisfies the assumptions of Section* 8.1, *and* n *is such that* c_n *is not near infinity or the point* a, *the Padé polynomials are unique for large* n, *and their normalization may be chosen so that*

i) $$p_j(z) \underset{n\to\infty}{\sim} \chi_j(z), \qquad j=1,\ 2,$$

uniformly for z *in any compact set not intersecting* S, *except for* z *near* c_n *in the case when* $\chi_j(c_n)=0$.

ii) $$p_j(z) \underset{n\to\infty}{=} \chi_{j+}(z) + \chi_{j-}(z) + o(1), \qquad j=1,\ 2,$$

uniformly for any compact subset of S *not containing the point* a.

iii) $$p_j(z) \underset{n\to\infty}{=} \chi_{j-}(z)\phi_0\left[((z-a)e^{-i\pi/3})^{3/2}\right] + \chi_{j+}(z)\phi_0\left[((z-a)e^{i\pi/3})^{3/2}\right] + o(1), \qquad j=1,\ 2,$$

for $|z-a|<\varepsilon_1$, *sufficiently small, fixed. The function* $\phi_0(x)$ *is defined in* (8.3.33).

The formula iii) holds for z on an arc of S assumed to leave a in the direction of the positive real axis. There are similar formulas for the other arcs of S. Each formula may be continued analytically through an angle $\pi/3$ on either side of the arc on which it applies.

CHAPTER 5: INTEGRAL EQUATION FOR REMAINDER FUNCTION

This chapter begins the derivation of the new results to be found in this thesis. As explained in Chapter 4, we are concerned with functions $f(z)$ that are expressible in the form

$$f(z) = f_0 + (2\pi i)^{-1}\int_S \omega(t)(t - z)^{-1}dt.$$

Here, S is a set of the class $\mathbb{M}$ described in Chapter 3 and $\omega(t)$ is a weight function satisfying certain conditions. In terms of the diagonal Padé polynomials $p_1(z)$, $p_2(z)$ of degree n, the remainder function $R(z)$ is defined as for example in (2.3.5) by

$$R(z) = p_1(z) + f(z)p_2(z).$$

In the special case when $S = L$, the line segment joining $z = -1, 1$, Nuttall [30] derived in 1988 a singular integral equation for $R(z)$. This equation was expressed with the help of the Riemann surface $\mathbb{R}$ consisting of two copies of $\mathbb{C}\backslash L$ attached across L. This surface has genus zero. When S is such that the corresponding surface has genus greater than zero, this approach does not appear to be fruitful. Consequently, our first step has been to derive a new version of the integral equation involving the restriction of $R(z)$ to S. For three different examples of S, this equation is derived below.

Throughout this chapter, we assume that $\omega(z)$ is such that all the integrals and limits involved exist where necessary and that other operations are valid. This will be the case for the examples discussed in succeeding chapters.

5.1 Case of $S = L$

In this case, the resulting equation is equivalent to that of Nuttall [30], but a discussion of this simple case will be helpful to the understanding of the next two examples.

Taking the limit as $z \to L$ from either side, we obtain

$$\begin{aligned} R_+(z) &= p_1(z) + p_2(z)f_+(z) \\ R_-(z) &= p_1(z) + p_2(z)f_-(z) \end{aligned} \qquad z \in L. \tag{5.1.1}$$

Subtracting gives

$$R_+(z) - R_-(z) = p_2(z)\left[f_+(z) - f_-(z)\right], \quad z \in L. \tag{5.1.2}$$

With the definition $y(z)$ in (3.1.1), (5.1.2) may be rewritten as

$$y_+(z)R_+(z) + y_-(z)R_-(z) = p_2(z)\sigma(z), \qquad z \in L, \tag{5.1.3}$$

where

$$\begin{aligned} \sigma(z) &= y_+(z)\omega(z) \\ &= y_+(z)\left[f_+(z) - f_-(z)\right], \qquad z \in L. \end{aligned} \tag{5.1.4}$$

With the help of the function,

$$\rho(z) = y(z)R(z), \tag{5.1.5}$$

analytic in $\mathbb{C}\backslash L$, (5.1.3) takes the alternate forms

$$\begin{aligned} \rho_+(z) &= p_2(z)\sigma(z) - \rho_-(z) \\ \rho_-(z) &= p_2(z)\sigma(z) - \rho_+(z) \end{aligned} \qquad z \in L. \tag{5.1.6}$$

We think of this as an inhomogeneous Hilbert problem [20] for the functions $\rho(z)$, $p_2(z)$, with inhomogeneous terms $-\rho_-(z)$, $-\rho_+(z)$. To 'solve' we first find solutions $h(z)$, $\chi(z)$ of the homogeneous problem

$$\begin{aligned} h_+(z) &= \chi_-(z)\sigma(z) \\ h_-(z) &= \chi_+(z)\sigma(z) \end{aligned} \qquad z \in L, \tag{5.1.7}$$

where $h(z)$, $\chi(z)$ are analytic and non-zero in $\mathbb{C}\backslash L$ and near infinity obey $h(z) \sim cz^{-n}$, $\chi(z) \sim cz^{n}$. Their product is independent of z, and we take it to be unity. The forms of $h(z)$, $\chi(z)$ may be given explicitly and may be be found in (2.1.5), (2.1.4).

To proceed, we introduce functions $U(z)$, $Q(z)$, analytic in $\mathbb{C}\backslash L$, with $U(z)$, $Q(z) \sim$ const., as $z \to \infty$,

$$U(z) = \rho(z)/h(z)$$

(5.1.8) $\qquad z \in \mathbb{C}\backslash L,$

$$Q(z) = p_2(z)/\chi(z)$$

so that (5.1.6) becomes

$$U_+(z) = Q_-(z) - U_-(z)H(z)$$

(5.1.9) $\qquad z \in L.$

$$U_-(z) = Q_+(z) - U_+(z)H(z)^{-1}$$

Here

(5.1.10) $\qquad H(z) = h_-(z)/h_+(z), \qquad z \in L.$

Equations (5.1.9) may be decoupled by defining

$$W(z) = U(z) + Q(z)$$

(5.1.11) $\qquad z \in \mathbb{C}\backslash L,$

$$Z(z) = U(z) - Q(z)$$

so that

$$W_+(z) - W_-(z) = -U_-(z)H(z) + U_+(z)H(z)^{-1}$$

(5.1.12) $\qquad z \in L.$

$$Z_+(z) + Z_-(z) = -U_-(z)H(z) - U_+(z)H(z)^{-1}$$

Since $W(z)$, $Z(z)$ are bounded at infinity, the Plemelj formula [20] shows that

$$W(z)=C_0 + (2\pi i)^{-1}\int_L \left[-U_-(t)H(t)+U_+(t)H(t)^{-1}\right](t-z)^{-1}dt$$

(5.1.13) $z\in\mathbf{C}\backslash L$

$$Z(z)=(2\pi i)^{-1}y(z)\int_L \left[-U_-(t)H(t)-U_+(t)H(t)^{-1}\right]y_+(t)^{-1}(t-z)^{-1}dt$$

where C_0 is a constant which depends on normalization. An integral equation for $U(z)$, $z \in L$ results on using

(5.1.14) $$U(z) = (W(z) + Z(z))/2,$$

which follows from (5.1.11), so that

(5.1.15) $$U(z)=C_0/2 +(4\pi i)^{-1}\int_L \left[1 - y(z)y_+(t)^{-1}\right]H(t)^{-1}U_+(t)(t-z)^{-1}dt$$

$$-(4\pi i)^{-1}\int_L \left[1 - y(z)y_-(t)^{-1}\right]H(t)U_-(t)(t-z)^{-1}dt, \quad z\in\mathbf{C}\backslash L.$$

The integral equation follows on taking the limits $z \to L_+$, L_-. As before [30], this is a singular integral equation, but nevertheless it is helpful in deriving asymptotic results. An example of its use is given in Chapter 6.

In order to derive the asymptotic formula for the Padé polynomials, we will need the expression for $Q(z)$ corresponding to (5.1.15), which is given by

$$(5.1.16)\quad Q(z)=C_0/2 + (4\pi i)^{-1}\int_L \left[1 + y(z)y_+(t)^{-1}\right]H(t)^{-1}U_+(t)(t-z)^{-1}dt$$

$$-(4\pi i)^{-1}\int_L \left[1 + y(z)y_-(t)^{-1}\right]H(t)U_-(t)(t-z)^{-1}dt, \quad z\in\mathbb{C}\backslash L.$$

For later use we give the relation between the functions $h(z)$, $\chi(z)$ and the functions $\chi_j(z)$, $j=1, 2$, $R_0(z)$ used in the conjecture of Section 3.3. We define

$$\chi_2(z) = \chi(z)$$

$$(5.1.17)\qquad \chi_1(z) = -f(z)\chi(z)$$

$$R_0(z) = h(z)/y(z).$$

With these definitions, (3.3.3) is obvious and (3.3.4) follows because, for $z\in L$,

$$\chi_{1+}(z) + f_-(z)\chi_{2+}(z) = -f_+(z)\chi_+(z) + f_-(z)\chi_+(z)$$

$$= -\sigma(z)\chi_+(z)/y_+(z)$$

$$= -h_-(z)/y_+(z)$$

$$= R_{0-}(z). \qquad \text{(from (5.1.7))}$$

5.2 Case of $m = 2$

Now we suppose that S corresponds to a Riemann surface with four branch points. We shall discuss two cases later. The first is when S consists of two non-intersecting analytic arcs joining pairs of branch points (Chapter 7). In the second case, S has three arcs meeting at one of the branch points (Chapter 8). The derivation and structure of the integral equation is the same for both cases and is given below.

We follow the algebraic manipulations of Section 5.1 as far as (5.1.12). Note that L must now be replaced by S, that

$$(5.2.1) \qquad y(z)^2 = \prod_{j=1}^{4} (z - b_j)$$

and that $h(z) \sim cz^{-n+1}$. In addition, we now have, with a suitable choice of normalization,

$$(5.2.2) \qquad h(z)\chi(z) = (z - c_n), \qquad z \in \mathbf{C},$$

for some choice of c_n.

The formula for $h(z)$ is given in [24]. Except for the special cases when $c_n \in S$ or $c_n = \infty$, for which the arguments must be modified, it follows that one or other of $h(z)$, $\chi(z)$ has a simple zero at $z = c_n$. Thus either $U(z)$ or $Q(z)$ has a simple pole at $z = c_n$.

Assuming that $h(c_n) = 0$, in which case $U(z)$ has a simple pole at $z=c_n$, the solution of (5.1.12) is now

$$W(z)=C_0+A(z-c_n)^{-1}+(2\pi i)^{-1}\int_S\left[-U_-(t)H(t)+U_+(t)H(t)^{-1}\right](t-z)^{-1}dt,$$

(5.2.3) $z \in \mathbf{C}\backslash S$

$$Z(z)=A(z-c_n)^{-1}y(z)y(c_n)^{-1}$$

$$+(2\pi i)^{-1}y(z)\int_S\left[-U_-(t)H(t)-U_+(t)H(t)^{-1}\right]y_+(t)^{-1}(t-z)^{-1}dt.$$

where A is a constant.

Because Z(z) is bounded at infinity, it is found that

(5.2.4) $$A\,y(c_n)^{-1} = -(2\pi i)^{-1}\int_S\left[U_-(t)H(t) + U_+(t)H(t)^{-1}\right]y_+(t)^{-1}dt.$$

Thus, using (5.1.14), which follows from (5.1.11), we have

(5.2.5) $$U(z) = C_0/2$$

$$+(4\pi i)^{-1}\int_S\left[(y_+(t)-y(z))(t-z)^{-1}+(y(c_n)+y(z))(c_n-z)^{-1}\right]U_+(t)H(t)^{-1}y_+(t)^{-1}dt$$

$$-(4\pi i)^{-1}\int_S\left[(y_-(t)-y(z))(t-z)^{-1}+(y(c_n)+y(z))(c_n-z)^{-1}\right]U_-(t)H(t)y_-(t)^{-1}dt.$$

Again we take the limits $z \to S_+$, S_- to obtain the required integral equation.

The other alternative is that $\chi(c_n)=0$ and that $h(c_n)\neq 0$. In this case $Q(z)$ has a simple pole at $z=c_n$, and the solution of (5.1.12) now may be written as

$$W(z)=C_0+A(z-c_n)^{-1}+(2\pi i)^{-1}\int_S\left[-U_-(t)H(t)+U_+(t)H(t)^{-1}\right](t-z)^{-1}dt,$$

(5.2.6) $z\in\mathbb{C}\setminus S$

$$Z(z)=-A(z-c_n)^{-1}y(z)y(c_n)^{-1}$$

$$+(2\pi i)^{-1}y(z)\int_S\left[-U_-(t)H(t)-U_+(t)H(t)^{-1}\right]y_+(t)^{-1}(t-z)^{-1}dt.$$

The bound on $Z(z)$ at infinity shows that

(5.2.7) $$Ay(c_n)^{-1}=(2\pi i)^{-1}\int_S\left[U_-(t)H(t)+U_+(t)H(t)^{-1}\right]y_+(t)^{-1}dt.$$

The result is that

(5.2.8) $U(z) = C_0/2$

$$+(4\pi i)^{-1}\int_S\left[(y_+(t)-y(z))(t-z)^{-1}-(y(c_n)-y(z))(c_n-z)^{-1}\right]U_+(t)H(t)^{-1}y_+(t)^{-1}dt$$

$$-(4\pi i)^{-1}\int_S\left[(y_-(t)-y(z))(t-z)^{-1}-(y(c_n)-y(z))(c_n-z)^{-1}\right]U_-(t)H(t)y_-(t)^{-1}dt.$$

The formulas for $Q(z)$ corresponding to the two cases, $h(c_n)=0$, or $\chi(c_n)=0$, are respectively given by

(5.2.9) $Q(z) = C_0/2$

$$+(4\pi i)^{-1}\int_S \Big[(y_+(t)+y(z))(t-z)^{-1}+(y(c_n)-y(z))(c_n-z)^{-1}\Big]U_+(t)H(t)^{-1}y_+(t)^{-1}dt$$

$$-(4\pi i)^{-1}\int_S \Big[(y_-(t)+y(z))(t-z)^{-1}+(y(c_n)-y(z))(c_n-z)^{-1}\Big]U_-(t)H(t)y_-(t)^{-1}dt.$$

and

(5.2.10) $Q(z) = C_0/2$

$$+(4\pi i)^{-1}\int_S \Big[(y_+(t)+y(z))(t-z)^{-1}-(y(c_n)+y(z))(c_n-z)^{-1}\Big]U_+(t)H(t)^{-1}y_+(t)^{-1}dt$$

$$-(4\pi i)^{-1}\int_S \Big[(y_-(t)+y(z))(t-z)^{-1}-(y(c_n)+y(z))(c_n-z)^{-1}\Big]U_-(t)H(t)y_-(t)^{-1}dt.$$

The formulas giving $\chi_j(z)$, $j=1$, 2 and $R_0(z)$ are as in (5.1.17) but here $h(z)$, $\chi(z)$, $y(z)$ are as used in this section.

CHAPTER 6: BRANCH POINT NOT OF SQUARE ROOT TYPE

At a point, say b_1, where one arc of S ends, the function f(z) has a branch point. Previous proofs of convergence of Padé approximants and the strong asymptotics of Padé polynomials [23], [24], [26], [27] have assumed that such a branch point was dominantly of square root (or inverse square root) type.

In this chapter, we discuss an example of a class of functions f(z) with a branch point not of square root type. To make the analysis as simple as possible, we will make favourable assumptions about the other characteristics of f(z). In particular, we assume that the preferred set is $S_1 = L$, and that the non-square root branch point is at z=1.

In order to obtain the fundamental theorem of this chapter and the asymptotic formulas of the Padé polynomials near a non-square root singularity, a number of lemmas are essential. (For the convenience of readers, we will usually state the lemmas without proof). The necessary proofs are given in the last section of this chapter.

6.1 Assumptions

We start this section with a definition.

Definition 6.1 *For a given, fixed quantity $\delta>0$, we define the closed curve Γ_1 by*

$$\Gamma_1 = \left\{z\in\mathbf{C}:\ \mathrm{Re}\,\phi(z) = -\delta\right\}, \tag{6.1.1}$$

where $\phi(z)$ is given by (3.1.5) or (2.1.7). In fact, Γ_1 is an ellipse with foci $z=-1, +1$, which surrounds the line segment L. It intersects the positive real axis at a point $d>1$. We define the line segment Γ_2 and Γ_3 by

$$\Gamma_2 = \left\{z\in\mathbf{C}:\ \mathrm{Im}z = 0,\ 1\le z\le d\right\},$$

(6.1.2)

$$\Gamma_3 = \left\{z\in\mathbf{C}:\ \mathrm{Im}z = 0,\ -d\le z\le -1\right\},$$

and set $\Gamma = \Gamma_1 \cup \Gamma_2$. We also define $D_+ = \mathrm{int}\,\Gamma \cap (HP)^+$, where $\mathrm{int}\,\Gamma$ denotes the interior of Γ, and $(HP)^+$ means the upper half plane. Similarly we may define $D_- = \mathrm{int}\,\Gamma \cap (HP)^-$.

We now consider a function $f(z)$ with the form

$$f(z) = f_0 + (2\pi i)^{-1}\int_L y_+(t)^{-1}\sigma(t)(t-z)^{-1}dt, \tag{6.1.3}$$

where

$$L = \left\{z\in\mathbf{C}:\ z \text{ real},\ -1\le z\le 1\right\},$$

$$y(t) = (t^2 - 1)^{1/2}$$

are as described in previous chapters.

Assumption 6.2 *We assume that* $\sigma(z)=\xi(z)(z-1)^{\nu}$, *where the function* $\xi(z)$ *is analytic and non-zero in the interior of* Γ_1 *and continuous and non-zero on* Γ_1. *Here,* ν *is a real parameter satisfying* $0 < \nu < 1/2$, *and* $(z-1)^{\nu}$ *is analytic in the interior of* Γ *with value* $\exp(i\pi\nu)$ *at* $z=0$.

These assumptions imply that $f(z)$ has dominant behaviour $(z-1)^{\nu-1/2}$ near $z=1$ and $(z+1)^{-1/2}$ near $z=-1$. Our aim is to analyze the effect of the non-square root singularity at $z=1$ on the asymptotics of the Padé polynomials. For such a problem, a brief discussion has been given by Nuttall and Li [17], where a slightly different technique was used.

We now state the following lemma, which will be proved in Section 6.6.

Lemma 6.3 *With the Assumption 6.2, the function* $h(z)$, *analytic in* $\mathbb{C}\backslash L$, *may be continued analytically through* L *from* D_+ *into* D_- *to give* $h_2(z)$, $z \in D_-$, *and similarly through* L *from* D_- *into* D_+ *to give* $h_1(z)$, $z \in D_+$. *The functions* $h_1(z)$, $h_2(z)$ *are non-zero in* $\bar{D}_+$, $\bar{D}_-$ *respectively. We have*

$$(6.1.4) \qquad h_1(z) = h_2(z), \qquad z \text{ real}, \ -d \le z \le -1,$$

$$(6.1.5) \qquad h(z)/h_1(z) = \exp\left[i\pi\nu + 2n\phi(z)\right]\left[1 + O(|(z-1)^{1/2}\log(z-1)|)\right],$$

$$(6.1.6) \qquad h(z)/h_2(z) = \exp\left[-i\pi\nu + 2n\phi(z)\right]\left[1 + O(|(z-1)^{1/2}\log(z-1)|)\right],$$

where z *is real,* $z \ge 1$, $z \to 1$.

6.2 Distorted Integral Equation

The integral equation (5.1.15) involves a knowledge of the function $h(z)$, analytic for $z \in \mathbb{C} \setminus L$. In this case, the explicit formula for $h(z)$ is given by (2.1.5), and Lemma 6.3 shows that the function $H(z)$ defined in (5.1.10) for $z \in L$ has an analytic continuation into the interior of Γ, which is continuous on the boundary of Γ. In particular, for $z \in \Gamma_-$, we have

$$
(6.2.1) \qquad \begin{aligned} H_+(z) &= h_1(z)/h(z) \\ H_-(z) &= h(z)/h_2(z) \end{aligned}
$$

Now suppose that, in (5.1.15), z is outside Γ_1. We observe that the first part of the integrand,

$$
(6.2.2) \qquad \left[1 - y(z)y_+(t)^{-1}\right]H(t)^{-1}U_+(t)(t-z)^{-1}, \qquad t \in L,
$$

is the limit from the + side of L of a function of t analytic for $t \in D_+$. Thus this part of the integral in (5.1.15) may be rewritten as

$$
(6.2.3) \qquad \int_{\Gamma_1^+ \cup \Gamma_2 \cup \Gamma_3} \left[1 - y(z)y(t)^{-1}\right]\mathbb{H}(t)U(t)(t-z)^{-1}dt.
$$

where $\Gamma_1^+ = \bar{D}_+ \cap \Gamma_1$ and we similarly have $\Gamma_1^- = \bar{D}_- \cap \Gamma_1$. The function $\mathbb{H}(t)$ is given by

$$(6.2.4)\qquad \mathbb{H}(t) = -\begin{cases} h(t)/h_1(t), & t \in \Gamma_1^+ \\ h(t)/h_2(t), & t \in \Gamma_1^- \\ h(t)(h_1(t)^{-1}-h_2(t)^{-1}), & t \in \Gamma_2 \cup \Gamma_3. \end{cases}$$

A similar procedure may be applied to the other part of the integral, with the result that (5.1.15) may be transformed to

$$(6.2.5)\quad U(z) = C_0/2 + (4\pi i)^{-1}\int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3}\left[1 - y(z)y(t)^{-1}\right]\mathbb{H}(t)U(t)(t-z)^{-1}dt,$$

On account of (6.1.4) of Lemma 6.3, we have $\mathbb{H}(t)=0$, $t \in \Gamma_3$. Also, the integral above is analytic in z for all z except perhaps z = 1, so that we have, with the help of Lemma 6.3,

Lemma 6.4 *With the Assumption* 6.2, U(z) *satisfies the following integral equation*

$$(6.2.6)\quad U(z) = C_0/2 + (4\pi i)^{-1}\int_{\Gamma}\left[1 - y(z)y(t)^{-1}\right]\mathbb{H}(t)U(t)(t-z)^{-1}dt, \; z \neq 1.$$

For $t \in \Gamma_2$, *we have*

$$(6.2.7)\qquad \mathbb{H}(t) = 4\pi i \exp\left[2n\phi(t)\right]H_0(t)$$

where

$$H_0(t) = -(1/2)\lambda_\nu\left[1 + O(|(z-1)^{1/2}\log(z-1)|)\right], \quad t\to 1,\ t\in\Gamma_2, \tag{6.2.8}$$

and

$$\lambda_\nu = (\sin\nu\pi)/\pi. \tag{6.2.9}$$

6.3 Functional Analysis of Integral Equation

The equation (6.2.6) may be regarded as an integral equation for $U(z)$, $z\in\Gamma$. If it is solved, the solution may be inserted into (6.2.6) to give $U(z)$ for other values of z. In contrast to the original equation (5.1.15), (6.2.6) is singular only at the point $z = 1$.

To analyze the integral equation (6.2.6), we set up a Banach space $\mathbb{B}$, which must include any possible solution $U(z)$ that might be obtained from the Padé polynomial definition. We see from (6.2.6) that $U(z)$ is analytic in a neighbourhood of all points on Γ except $z = 1$. Near $z=1$, we have

$$U(z) \approx \text{const.}(z-1)^{\nu/2} \tag{6.3.1}$$

Thus the space

$$\mathbb{B} = \Big\{U(t): t\in\Gamma,\ U(t)\ \text{continuous except at}\ t=1, \tag{6.3.2}$$

$$\|U\| = \sup_{t\in\Gamma}\left|U(t)(t-1)^{\mu/2}\right|\Big\},$$

where $0 < \nu < \mu < 1/2$, is certainly adequate.

The linear operator $\mathbb{K}$, dependent on n, associated with (6.2.6) is defined by $Y = \mathbb{K}U$

$$Y(z) = \int_{\Gamma} K(z,t)U(t)dt, \tag{6.3.3}$$

where

$$K(z,t) = (4\pi i)^{-1}\left[1 - y(z)y(t)^{-1}\right]H(t)(t-z)^{-1}, \qquad z,\ t\in\Gamma. \tag{6.3.4}$$

Thus (6.2.6) may be written more abstractly as

$$U = C_0/2 + \mathbb{K}U \tag{6.3.5}$$

Later, we prove the following lemma

Lemma 6.5 *The operator* $\mathbb{K}: \mathbb{B} \rightarrow \mathbb{B}$, *is bounded, and there exists a value* n_0 *and a quantity* $k < 1$ *such that, for all* $n > n_0$,

$$\|\mathbb{K}\| \le k. \tag{6.3.6}$$

Since we have $C_0 \neq 0$, it follows immediately from standard results in functional analysis [15], that, there is, for all $n > n_0$, a unique solution of (6.2.6) in $\mathbb{B}$. This means that the Padé polynomials are unique up to a constant factor, and we choose their normalization so that $C_0 = 2$. Therefore (6.2.6) may be written as

$$(6.3.7)\qquad U(z) = 1 + (4\pi i)^{-1}\int_{\Gamma}\left[1 - y(z)y(t)^{-1}\right]H(t)U(t)(t-z)^{-1}dt,\ z\neq 1.$$

It follows that, for $n > n_0$, the solution of (6.3.7) satisfies

$$(6.3.8)\qquad \|U\| \le (1 - k)^{-1}\|1\|.$$

We are led to the fundamental theorem of this chapter.

Theorem 6.6 *With the Assumption 6.2, there exists a value* n_0, *such that, for all* $n > n_0$, *the remainder function* $R(z)$ *and the Padé polynomials* $p_j(z)$, $j=1,2$, *are unique up to a constant factor. For any given* $\varepsilon_1 > 0$, *when* $|z-1| > \varepsilon_1$, *the asymptotic form of* $R(z)$ *is uniformly given by*

$$(6.3.9)\qquad R(z) = y(z)^{-1}h(z)(1 + o(1)),\qquad n \to \infty.$$

Proof. The uniqueness of $U(z)$ implies the uniqueness of $R(z)$ and the Padé polynomials.

From (6.3.8) we have, for $n>n_0$,

$$(6.3.10)\qquad |U(t)| \le (1-k)^{-1}|t-1|^{-\mu/2},\qquad t \in \Gamma.$$

Thus, from (6.3.7),

$$|U(z)-1| \le \text{const.}\int_{\Gamma} |K(z,t)|\,|t-1|^{-\mu/2}|dt|. \tag{6.3.11}$$

From (6.3.4),(6.2.4) and (6.2.7) it follows that, uniformly for $|z-1| > \varepsilon_1$,

$$|K(z,t)| \le \text{const.}|t-1|^{-1/2}\exp\Big[2n\mathrm{Re}\phi(t)\Big], \qquad t\in\Gamma. \tag{6.3.12}$$

Now $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\mathrm{Re}\phi(t) = -\delta$, $t\in\Gamma_1$, so that the contribution to (6.3.11) from Γ_1 is bounded by $\exp(-2n\delta)$. For $t\in\Gamma_2$, $\mathrm{Re}\phi(t)$ decreases as t increases from 1 and near $t=1$,

$$\mathrm{Re}\phi(t) = -(t-1)^{1/2}\Big[2^{1/2} + O(t-1)\Big]. \tag{6.3.13}$$

Therefore this part of the integral is dominated by

$$\text{const.}\int_1^d (t-1)^{-(1+\mu)/2}\exp(-2^{3/2}n(t-1)^{1/2})dt = O(n^{-1+\mu}), \quad n\to\infty. \tag{6.3.14}$$

Thus the asymptotic form of $U(z)$, for $|z-1|>\varepsilon_1$, is uniformly given by

$$U(z) = 1 + o(1), \qquad n\to\infty. \tag{6.3.15}$$

The result follows since

$$R(z) = y(z)^{-1}h(z)U(z). \tag{6.3.16}$$ ■

Remark. With the help of results obtained in Section 6.4, the region of validity of (6.3.9) could be extended and the error bound decreased.

6.4 Form of Solution Near z=1

We have found an approximate form of the solution $U(z)$ of (6.3.7) valid for large n everywhere except near the point $z = 1$. The next task is to find an approximation to describe $U(z)$ near $z=1$. In essence this is done by introducing an operator K^*, chosen so that the kernel $K^*(z,t)$ approximates $K(z,t)$ when z, t are close to 1. We show that the solution of the equation (6.3.7) with K replaced by K^* is close to $U(z)$ when z is near 1. In this section we give the motivation and the results. The details of the proofs will be found in Section 6.6.

As remarked above, we are interested in the form of the kernel of (6.3.4) when z, $t \in \Gamma_2$ are both near to 1. It is convenient to replace these variables by x, w given by

$$\begin{aligned} x &= (z - 1)^{1/2} \\ w &= (t - 1)^{1/2} \end{aligned} \tag{6.4.1}$$

Using (6.2.7), (6.2.8), (2.1.7) and approximating where appropriate for small x, w, we find that the integral operator of (6.3.4) is transformed by

$$(6.4.2)\qquad (4\pi i)^{-1}\int_1^d \left[1 - y(z)y(t)^{-1}\right] H(t)^{*}(t-z)^{-1}dt \to -\lambda_\nu \int_0^{x_0} e^{-Nw}(x+w)^{-1}dw,$$

where and $x_0 = (d-1)^{1/2}$ and $N = 2^{3/2}n$.

For the convenience of analysis, we extend the upper limit of the w-integration to infinity and define the linear operator $\mathbf{V}$, acting an function $v(x)$, $0 \le x < \infty$, by $q = \mathbf{V}v$, where

$$(6.4.3)\qquad q(x) = \int_0^\infty V(x,w)v(w)dw,$$

with

$$(6.4.4)\qquad V(x,w) = -\lambda_\nu e^{-Nw}(x + w)^{-1}.$$

A Banach space $\mathbf{B}_1$, suitable for the analysis of $\mathbf{V}$, is

$$(6.4.5)\qquad \mathbf{B}_1 = \left\{ v(x): 0 \le x < \infty,\ v(x) \text{ continuous except at } x=0, \right.$$

$$\left. \|v\|_1 = \sup_{0 \le x < \infty} \left| v(x)x^{\mu} \right| \right\}.$$

The following lemma holds.

Lemma 6.7 *There exists* $k_1 < 1$ *such that the operator* $\mathbf{V}:\mathbf{B}_1 \to \mathbf{B}_1$, *is bounded with* $\|\mathbf{V}\|_1 < k_1$ *for all* $N \ge 1$.

In terms of the modified Bessel function [32], we define two functions which will be needed later. They are

$$\phi_1(x) = \pi^{-1/2}N^{1/2}x^{1/2}e^{Nx/2}K_{\nu-1/2}(Nx/2) \tag{6.4.6}$$

and

$$\phi_2(x) = \pi^{-1/2}N^{1/2}x^{3/2}e^{Nx/2}K_{\nu+1/2}(Nx/2). \tag{6.4.7}$$

These functions are analytic in the complex plane cut along the negative real axis, and

$$\phi_1(x) \underset{|x|\to\infty}{\sim} 1, \tag{6.4.8}$$

and

$$\phi_2(x) \underset{|x|\to\infty}{\sim} x + C^*, \tag{6.4.9}$$

where $C^* = (4\nu^2-1)/N$.

Lemma 6.8 *The functions $\phi_1(x)$ and $\phi_2(x)$ satisfy the following integral equations*

$$\phi_1(x) = 1 + \int_0^\infty V(x,w)\phi_1(w)\,dw, \qquad 0 \le x < \infty, \tag{6.4.10}$$

and

$$\phi_2(x) = x + C^* + \int_0^\infty V(x,w)\phi_2(w)\,dw, \qquad 0 \le x < \infty. \tag{6.4.11}$$

Lemma 6.9 *For any function $\gamma(x)$, with $\gamma(x)/(1 + x^2) \in B_1$, the solution of the integral equation*

$$\psi(x) = \gamma(x) + \int_0^\infty V(x,w)\psi(x)\,dw, \tag{6.4.12}$$

is given by

$$\psi(x) = \gamma(x) + \int_0^\infty g(x,w)\gamma(w)\,dw, \qquad 0 \le x < \infty, \tag{6.4.13}$$

where

$$g(x,w) = -\lambda_\nu e^{-Nw}\Big[\phi_1(x)\phi_2(w) - \phi_2(x)\phi_1(w)\Big](w^2 - x^2)^{-1}. \tag{6.4.14}$$

To rigorously examine the behaviour of $U(z)$ near $z = 1$, we rewrite the integral equation (6.3.7) as

$$U(z) = 1 + \int_{\Gamma_1} K(z,t)U(t)\,dt + \int_{\Gamma_2} K(z,t)U(t)\,dt, \qquad z \in \Gamma_2. \tag{6.4.15}$$

We define

$$u(x) = U(z), \qquad z \in \Gamma_2. \tag{6.4.16}$$

Using (6.4.1) we set

$$2wK(z,t) = V(x,w) + V_1(x,w) + V_2(x,w) + V_3(x,w), \qquad z,\ t\in\Gamma_2, \tag{6.4.17}$$

where

$$V_1(x,w) = 2e^{2n\phi(t)}H_0(t)\left\{w(t - z)^{-1}\left[1 - y(z)y(t)^{-1}\right]-(x + w)^{-1}\right\}, \tag{6.4.18}$$

$$V_2(x,w) = 2H_0(t)(x + w)^{-1}\left[e^{2n\phi(t)} - e^{-Nw}\right], \tag{6.4.19}$$

$$V_3(x,w) = (x + w)^{-1}e^{-Nw}\left[2H_0(t) + \lambda_\nu\right]. \tag{6.4.20}$$

Thus, (6.4.15) may be written as

$$u(x) = 1 + \tau(x) + \int_0^{x_0} V(x,w)u(w)\,dw, \qquad 0\le x\le x_0, \tag{6.4.21}$$

where

$$\tau(x) = \int_{\Gamma_1} K(z,t)U(t)\,dt + \sum_{j=1}^{3}\int_0^{x_0} V_j(x,w)u(w)\,dw, \qquad z\in\Gamma_2. \tag{6.4.22}$$

If we arbitrarily define

$$\tau(x) = \tau(x_0), \qquad \text{for } x > x_0, \tag{6.4.23}$$

then (6.4.21) may be extended to read

$$u(x) = 1 + b(x) + \int_0^\infty V(x,w)u(w)\,dw, \qquad 0 \le x < \infty, \tag{6.4.24}$$

where

$$b(x) = \tau(x) - \int_{x_0}^\infty V(x,w)u(w)\,dw, \qquad 0 \le x < \infty. \tag{6.4.25}$$

We regard (6.4.23), (6.4.25) and (6.4.24) as providing a definition of $u(x)$, $x_0 \le x < \infty$.

The main result follows from two lemmas, proved in Section 6.6. They are

Lemma 6.10.

i) *If* $\|U\| < C$, *then there exists a constant* C_1, *which tends to* 0 *as* $n \to \infty$, *such that*

$$|b(x)| < CC_1, \qquad 0 \le x < \infty. \tag{6.4.26}$$

ii) *If* $|U(z)| < C$, $z \in \Gamma$, *then there exists a constant* C_2, *and a function*

$C_3(x)$, $0 \le x < \infty$, *such that*

$$(6.4.27) \qquad b(x) = C_2 + \left[x/(1+x)\right]^{\nu} C_3(x), \qquad 0 \le x < \infty.$$

For $0 \le x < \infty$, $|C_3(x)| < C_4$, *where* C_2, $C_4 \to 0$, *as* $n \to \omega$.

Lemma 6.11

i) If $|b(x)| < C_8$, $0 \le x < \infty$, *then the solution of* (6.4.24) *satisfies*

$$(6.4.28) \qquad |u(x) - \phi_1(x)| < C_8 C_5, \qquad 0 \le x < \infty,$$

where C_5 *is a constant independent of* n.

ii) If b(x) *satisfies* (6.4.27), *then the solution of* (6.4.24) *satisfies*

$$(6.4.29) \qquad |u(x) - \phi_1(x)| < C_6 \left[Nx/(1+Nx)\right]^{\nu}, \qquad 0 \le x < \infty,$$

where $C_o \to 0$ *as* $n \to \infty$.

With the help of these lemmas, the main result about the behaviour of U(z) near to z = 1 can be proved.

Theorem 6.12 *For* $|\arg(z-1)| \le 2\pi$, *and* $|z-1| \le \varepsilon_1$, *suitably small, fixed, the asymptotic form of* U(z) *is uniformly given by*

$$(6.4.30) \qquad U(z) = \phi_1((z-1)^{1/2})(1 + o(1)), \qquad n \to \infty.$$

Proof. Equation (6.3.8) shows that, for large n, there exists a constant C, independent of n, such that $\|U\|<C$, and Lemma 6.10i) leads to (6.4.26). From this, Lemma 6.11i) gives (6.4.28). Since $|\phi_1(x)|$ is bounded by a constant, independent of x and n, $0\le x<\infty$, the requirements of Lemma 6.10ii) hold, and we deduce (6.4.27). From Lemma 6.11ii) this gives (6.4.29), which is equivalent to (6.4.30) for z real, and $z>1$.

It is not difficult to extend the analysis of $u(x)$ given by (6.6.50) to complex values of x near $x = 0$. We note that to obtain results for $\arg x = \pm\pi$ (i.e. $|\arg(z-1)| = 2\pi$) it is necessary to distort slightly the contour of integration in integrals such as the one in (6.6.50) and the one for $b(x)$ which gives $C_3(x)$. ∎

6.5 Asymptotic Formulas for Padé Polynomials

The polynomial $p_2(z)$ is given in terms of $Q(z)$ in (5.1.8), so that

(6.5.1) $$p_2(z) = Q(z)\chi(z),$$

and $Q(z)$ is given in terms of $U(z)$ by (5.1.16). Alternatively, for $z\in L$, we may eliminate $p_1(z)$ from (5.1.1) to give

(6.5.2) $$\begin{aligned} p_2(z) &= (R_+(z) - R_-(z))/(f_+(z) - f_-(z)) \\ &= (h_+(z)U_+(z) + h_-(z)U_-(z))/\sigma(z) \\ &= \chi_-(z)U_+(z) + \chi_+(z)U_-(z) \end{aligned}$$

Similarly, it is found that

$$(6.5.3) \qquad p_1(z) = -f_-(z)\chi_-(z)U_+(z) - f_+(z)\chi_+(z)U_-(z),$$

from (5.1.4), (5.1.7), (5.1.8). These equations may be continued into the region of analyticity of $\sigma(z)$, the interior of Γ.

Just as for U(z) given by (5.1.15), the equation (5.1.16) for Q(z) may be rewritten in analogy with (6.2.6) to give

$$(6.5.4) \qquad Q(z) = C_0/2 + (4\pi i)^{-1}\int_\Gamma \left[1 + y(z)y(t)^{-1}\right]H(t)U(t)(t-z)^{-1}dt$$

$$+ h(z)U(z)/h_c(z), \qquad z\notin\Gamma.$$

The last term must be omitted if z is outside Γ. If z is inside Γ, $h_c(z)$ means $h_1(z)$ if $z \in D_+$, $h_2(z)$ if $z \in D_-$.

Part i) of Theorem 4.1 is proved by choosing $\delta > 0$ so that z is outside Γ and substituting the results of Theorems 6.6, 6.12 for U(z) into (6.5.4). To obtain the form of $p_1(z)$, solve (2.3.5) to give

$$(6.5.5) \qquad p_1(z) = R(z) - f(z)p_2(z)$$

and use $R(z) << p_2(z) \approx \chi_2(z)$, $z\notin L$, $n\to\infty$ and $\chi_1(z) = -f(z)\chi_2(z)$ from (5.1.17).

Part ii) is obtained immediately on using (6.5.2), (6.5.3).

To prove part iii), substitute the behaviour of U(z) given by

Theorem 6.12 into (6.5.2), (6.5.3).

It is interesting to check this result against the well-known formula for the asymptotic behaviour of Jacobi polynomials, found in Szegö ([45], Theorem 8.21.12). Near z=1, $\chi_2(z)=\chi(z)$ may be approximated by const.$(z-1)^{-\nu/2}\exp(-n\phi(z))$, so that, with $z = 1 - v^2$, v real, v > 0, we have from Theorem 4.1 iii) or (6.5.2), for small v,

$$(6.5.6) \qquad p_2(z) = \chi_-(z)U_+(z) + \chi_+(z)U_-(z)$$

$$\approx \text{const.}\left[(-iv)^{-\nu}e^{-iN\nu/2}\phi_1(iv) + (iv)^{-\nu}e^{iN\nu/2}\phi_1(-iv)\right].$$

We insert (6.4.6) for $\phi_1(x)$ and use the formulas

$$(6.5.7) \qquad K_\mu(x) = \pi(I_{-\mu}(x) - I_\mu(x))/(2\sin\mu\pi),$$

$$(6.5.8) \qquad I_\mu(ix) = e^{i\pi\mu/2}J_\mu(-x),$$

and

$$(6.5.9) \qquad J_\mu(-x) = e^{i\pi\mu}J_\mu(x)$$

to give

$$(6.5.10) \qquad p_2(z) \approx \text{const.}v^{-\nu+1/2}J_{\nu-1/2}(Nv/2)$$

which is equivalent to Szegö's result.

6.6 Proofs of Lemmas

Proof of Lemma 6.3 We give here only a proof of (6.1.5) since (6.1.6) is quite similar and (6.1.4) is obvious. If we write

$$(6.6.1) \qquad I(z) = \int_L y_+(t)^{-1} \log\sigma(t)(t-z)^{-1}dt.$$

then, with the definition y(z) of (3.1.1), (2.1.5) is equivalent to

$$(6.6.2) \qquad h(z) = \exp\left[(2\pi i)^{-1}y(z)I(z)\right]\exp\left[n\phi(z)\right].$$

Thus, it follows that

$$(6.6.3) \qquad h_+(z)/h_-(z) = \exp\left\{(2\pi i)^{-1}\left[y_+(z)I_+(z) - y_-(z)I_-(z)\right]\right\}$$

$$\cdot\exp\left\{n\left[\phi_+(z) - \phi_-(z)\right]\right\}, \qquad z\in L.$$

Note that

$$(6.6.4) \qquad \begin{aligned} y_+(z) &= - y_-(z) \\ \phi_+(z) &= - \phi_-(z) \end{aligned} \qquad z\in L,$$

which leads to

$$(6.6.5)\quad h_{+}(z)/h_{-}(z) = \exp\left\{(2\pi i)^{-1}y_{+}(z)\left[I_{+}(z)+I_{-}(z)\right]\right\}\exp\left[2n\phi_{+}(z)\right],\quad z\in L.$$

The Plemelj formula [20] shows that

$$(6.6.6)\qquad I_{+}(z) - I_{-}(z) = 2\pi i y_{+}(z)^{-1}\log\sigma(z),\qquad z\in L.$$

Substituting (6.6.6) into (6.6.5) leads to

$$(6.6.7)\quad h_{+}(z)/h_{-}(z) = \exp\left[(\pi i)^{-1}y_{+}(z)I_{+}(z)-\log\sigma(z)\right]\exp\left[2n\phi_{+}(z)\right],\quad z\in L.$$

Then, the analytic continuation of (6.6.7) into D_{+} gives

$$(6.6.8)\quad h(z)/h_{+}(z) = \exp\left[(\pi i)^{-1}y(z)I(z) - \log\sigma(z)\right]\exp\left[2n\phi(z)\right],\quad z\in D_{+}.$$

From Assumption 6.2, we have $\sigma(t) = \xi(t)(t-1)^{\nu}$, which means that $I(z)$ may be written as

$$(6.6.9)\quad I(z) = \int_L y_{+}^{-1}(t)\log\xi(t)(t-z)^{-1}dt + \nu\int_L y_{+}^{-1}(t)\log(t-1)(t-z)^{-1}dt.$$

After transforming through

$$(6.6.10)\qquad \begin{aligned} z' &= z-1 \\ t' &= t-1, \end{aligned}$$

then, (A1.1) and (A1.2) in Appendix 1 show that

$$(6.6.11)\qquad (\pi i)^{-1}y(z)I(z) = \log\xi(z) + \nu\log(z-1) + O(|(z-1)^{1/2}\log(z-1)|),$$

where z is real, $z\geq 1$, $z\to 1$.

Substitution (6.6.11) into (6.6.8) proves the required result. ∎

Proof of Lemma 6.4 A combination of (6.2.5) and Lemma 6.3 leads to Lemma 6.4. ∎

Proof of Lemma 6.5 Suppose that $U(z)$, $z\in\Gamma$, corresponds to a function $U\in B$ with $\|U\|=1$. Then, if

$$(6.6.12)\qquad Y(z) = \int_{\Gamma} K(z,t)U(t)\,dt,$$

we must show that, for large n,

$$(6.6.13)\qquad \sup_{t\in\Gamma}\left|Y(z)(z-1)^{\mu/2}\right| \leq k < 1.$$

For $z\in\Gamma_2$, this is done with the help of the analysis of Lemma 6.10 on $\tau(x)$. Thus again using the transformations (6.4.1)

$$z = x^2 + 1,$$
$$t = w^2 + 1,$$

and (6.4.16) $\qquad U(z) = u(x),$

we have

$$(6.6.14) \qquad Y(z) = \int_0^{x_0} V(x,w)u(w)dw + \tau(x), \qquad z\in\Gamma_2.$$

Lemma 6.10 i) implies that

$$(6.6.15) \qquad \sup_{z\in\Gamma_2} \left|\tau(x)(z-1)^{\mu/2}\right| \to 0, \qquad \text{as } n\to\infty.$$

Also Lemma 6.7 shows that

$$(6.6.16) \qquad \sup_{z\in\Gamma_2} \left|(z-1)^{\mu/2}\int_0^{x_0} V(x,w)u(w)dw\right| \le k_1 \sup_{z\in\Gamma_2} \left|(z-1)^{\mu/2}U(z)\right| \le k_2 < 1.$$

Thus, as far as $z\in\Gamma_2$ is concerned, (6.6.12) holds for large n. The rest of the proof is straightforward. ∎

Proof of Lemma 6.7 We suppose that $v\in B_1$ with $\|v\|_1=1$, which means that

$$(6.6.17) \qquad x^{\mu}|v(x)| \le 1, \qquad 0\le x<\infty.$$

If T=Uv, we must consider

$$(6.6.18) \qquad \|T\|_1 = \sup_{0\le x<\infty} \left|x^{\mu}T(x)\right|.$$

From (6.6.17), we have

$$x^{\mu}\left|T(x)\right| \leq \lambda_{\nu} x^{\mu} \int_0^{\infty} e^{-Nw} w^{-\mu}(x+w)^{-1} dw, \qquad 0 \leq x < \infty. \tag{6.6.19}$$

With w=xs, this gives

$$\begin{aligned} x^{\mu}\left|T(x)\right| &\leq \lambda_{\nu} \int_0^{\infty} e^{-Nxs} s^{-\mu}(1+s)^{-1} ds, \\ &\leq \lambda_{\nu} \int_0^{\infty} s^{-\mu}(1+s)^{-1} ds, \\ &\leq \lambda_{\nu}\lambda_{\mu}^{-1} \\ &< 1, \qquad 0 \leq x < \infty, \end{aligned} \tag{6.6.20}$$

and the Lemma is proved. ■

Remark. It may be shown that $\|U\|_1 = \lambda_{\nu}\lambda_{\mu}^{-1}$.

Proof of Lemma 6.8 The formula for analytic continuation of the modified Bessel function [32] shows that

$$(6.6.21) \qquad \lim_{\varepsilon\to 0+}\left[\phi_1(-(x+i\varepsilon))-\phi_1(-(x-i\varepsilon))\right] = 2i\sin\nu\pi\ e^{-Nx}\phi_1(x), \quad 0<x<\infty.$$

The asymptotic behaviour of $\phi_1(x)$ together with the Plemelj formula demonstrate (6.4.10). The proof of (6.4.11) is similar. ∎

Proof of Lemma 6.9 We prove this result by substituting the formula (6.4.13) into the right-hand side of (6.4.12) to give

$$(6.6.22) \quad \text{RHS of } (6.4.12)=\gamma(x)+\int_0^\infty V(x,w)\gamma(w)\,dw + \int_0^\infty V(x,w)\,dw\int_0^\infty g(w,s)\gamma(s)\,ds.$$

Interchanging the order of integration in the last term gives

$$(6.6.23) \qquad I(x) = \int_0^\infty V(x,w)\,dw\int_0^\infty g(w,s)\gamma(s)\,ds$$

$$= \lambda_\nu^2 \int_0^\infty \gamma(s)e^{-Ns}ds \int_0^\infty e^{-Nw}\left[\phi_1(w)\phi_2(s)-\phi_2(w)\phi_1(s)\right]$$

$$\cdot (x+w)^{-1}(s^2-w^2)^{-1}dw.$$

A partial fraction decomposition gives

$$(6.6.24) \quad (x+w)^{-1}(s^2-w^2)^{-1} = (2s)^{-1}\left[(s+w)^{-1}(x-s)^{-1}-(x+w)^{-1}(x-s)^{-1}\right]$$

$$+ (2s)^{-1}\left[(x+w)^{-1}(s+x)^{-1}+(s-w)^{-1}(s+x)^{-1}\right].$$

This is inserted into (6.6.23), with the help of (6.4.10) and (6.4.11) to give

$$I(x) = -\lambda_{\nu}\int_0^{\infty} \gamma(s)e^{-Ns}(2s)^{-1}\Omega(x,s)\,ds \tag{6.6.25}$$

where

$$\begin{aligned}\Omega(x,s) = {} & (x-s)^{-1}\Bigg\{-\phi_2(s)+\phi_1(s)(s+C^*)-\Big[\phi(x)\phi_2(s)-\phi_2(x)\phi_1(s)\Big]\\ & \qquad +\phi_2(s)-\phi_1(s)(x+C^*)\Big]\Bigg\}\\ & +(x+s)^{-1}\Bigg\{\Big[\phi_1(x)\phi_2(s)-\phi_2(x)\phi_1(s)\Big]-\phi_2(s)+\phi_1(s)(x+C^*)\\ & \qquad -\Big[\phi_1(-s)\phi_2(s)-\phi_2(-s)\phi_1(s)\Big]+\phi_2(s)+\phi_1(s)(s-C^*)\Big]\Bigg\},\end{aligned} \tag{6.6.26}$$

where $\phi_1(-s)$, $\phi_2(-s)$ mean the limit from above the negative real axis. With the help of the identity [32]

$$\phi_1(-s)\phi_2(s)-\phi_2(-s)\phi_1(s) = 2s \tag{6.6.27}$$

the expression for $I(x)$ may be simplified to

$$(6.6.28) \qquad I(x) = -\lambda_\nu \int_0^\infty e^{-Ns}\Big[\phi_1(x)\phi_2(s)-\phi_2(x)\phi_1(s)\Big](s^2-x^2)^{-1}\gamma(s)\,ds$$

$$+ \lambda_\nu \int_0^\infty e^{-Ns}(x+s)^{-1}\gamma(s)\,ds$$

$$= \int_0^\infty g(w,s)\gamma(s)\,ds - \int_0^\infty V(x,s)\gamma(s)\,ds.$$

Thus

$$(6.6.29) \qquad \text{RHS of } (6.4.12) = \gamma(s) + \int_0^\infty g(x,s)\gamma(s)\,ds,$$

and the lemma is proved. ∎

Proof of Lemma 6.10 The five contributions to $b(x)$ have to be treated separately.

$\int_{\Gamma} K(z,t)U(t)\,dt$ — use the fact that $|K(z,t)| < \text{const}.e^{-2n\delta}$,

for $t\in\Gamma_1$, $z\in\Gamma_2$.

$\int_0^{x_0} V_1(x,w)u(w)dw$ — use the mean value theorem to show that

$$(6.6.30) \qquad \left| w(t-z)^{-1}\left[1- y(z)y(t)^{-1}\right]-(x+w)^{-1} \right| \leq x\theta(x+w)^{\,}, \qquad 0\leq x,\ w\leq x ,$$

where θ lies between x, w.

$\int_0^{x_0} V_2(x,w)u(w)dw$ — from (3.1.5) we have

$$(6.6.31) \qquad \phi(t) = -2^{\,}w + O(w^{\,}), \qquad w\to 0, \qquad 0\leq w\leq x .$$

Thus

$$(6.6.32) \qquad |e^{-n\phi(t)}-e^{-Nw}| < \text{const}.e^{-Nw/2}N^3w^3, \qquad 0 \leq w < N^{\,},$$

$$< \text{const}.e^{-Nw\,}, \qquad N^{\,} < w < x .$$

$\int_0^{x} V_{\,}(x,w)u(w)dw$ — use (6.2.8) to show that

$$(6.6.33) \qquad |2H_{\,}(t) + \lambda| < \text{const}.w^{\alpha}, \qquad 0\leq w< x ,$$

where $\nu+\varepsilon<\alpha<1/2$.

For part i) we have $|u(w)| < \text{const}.w^{(\nu+\varepsilon)}$. Thus the bound on the contribution to $b(x)$ involves

$$(6.6.34)\qquad \int_0^{x_0} e^{-Nw}(x+w)^{-1}w^{\alpha-\nu-\varepsilon}\,dw < \int_0^{\infty} e^{-Nw}\, w^{\alpha-\nu-\varepsilon-1}\,dw \to 0, \quad \text{as } n \to \infty.$$

For part ii) we write

$$(6.6.35)\qquad \int_0^{x_0} V_3(x,w)u(w)\,dw = \int_0^{x_0} V_3(0,w)u(w)\,dw - x\int_0^{x_0} w^{-1}V_3(x,w)u(w)\,dw.$$

With $|u(x)|<\text{const}$, $0\le x\le x_0$, a bound on the first integral is

$$(6.6.36)\qquad \text{const.}\int_0^{x_0} e^{-Nw}w^{\alpha-1}\,dw \to 0, \qquad \text{as } N \to \infty.$$

With $\nu<\beta<\alpha$, a bound on the second term involves

$$(6.6.37)\qquad x\int_0^{x_0} e^{-Nw}(x+w)^{-1}w^{\alpha-1}\,dw = x^{\beta}\int_0^{x_0} e^{-Nw}\left[x/(x+w)\right]^{1-\beta}\left[w/(x+w)\right]^{\beta} w^{\alpha-\beta-1}\,dw$$

$$< x^{\beta}\int_0^{x_0} e^{-Nw}w^{\alpha-\beta-1}\,dw,$$

and the result follows since the integral $\to 0$ as $N \to \infty$.

$\int_{x_0}^{\infty} V(x,w)u(w)\,dw$ — the analysis is easy. ∎

Proof of Lemma 6.11 From Lemmas 6.8, 6.9, the solution of (6.4.24) is given by

$$(6.6.38) \qquad u(x) = \phi_1(x) + b(x) + \int_0^\infty g(x,w)b(w)\,dw, \qquad 0 \le x < \infty.$$

To analyze the integral in this expression, it is convenient to change variables as follows

$$(6.6.39) \qquad \begin{aligned} X &= Nx \\ W &= Nw \end{aligned}$$

and

$$(6.6.40) \qquad \Phi_1(X) = \phi_1(x),$$

$$(6.6.41) \qquad \Phi_2(X) = N\phi_2(x).$$

Then, it follows that

$$(6.6.42) \quad \int_0^\infty g(x,w)b(w)\,dw = -\lambda_\nu \int_0^\infty e^{-W}\Big[\Phi_1(X)\Phi_2(W) - \Phi_2(X)\Phi_1(W)\Big](W^{\cdot} - X^{\cdot})^{\cdot}\, b(w)\,dW.$$

For $0 \le X,\ W < \infty$, we use the bounds

$$(6.6.43) \qquad |\Phi_1(W)| < C_7\left[W/(1 + W)\right]^{\nu},$$

$$(6.6.44) \qquad |\Phi_2(W)| < C_7\left[W/(1 + W)\right]^{1-\nu}(1+W),$$

$$(6.6.45) \qquad \Phi_1(X)\Phi_2(W) - \Phi_2(X)\Phi_1(W) = (W-X)\left[\Phi_1(X)\Phi_2'(\theta) - \Phi_2(X)\Phi_1'(\theta)\right],$$

where θ is between X, W, and

$$(6.6.46) \qquad |\Phi_1'(W)| < C_7\left[W/(1 + W)\right]^{\nu-1},$$

$$(6.6.47) \qquad |\Phi_2'(W)| < C_7\left[W/(1 + W)\right]^{-\nu}.$$

These bounds are used as appropriate after splitting the integral in (6.6.42) in three parts

$$(6.6.48) \qquad \int_0^{X/2} + \int_{X/2}^{3X/2} + \int_{3X/2}^{\infty} dW$$

to give

$$(6.6.49) \qquad \left|\int_0^{\infty} g(x,w)b(w)dw\right| <$$

$$C_7\int_0^{X/2} e^{-W}\left[X^{\nu}W^{1-\nu} + X^{1-\nu}(1+X)^{\nu}W^{\nu}\right] X^{-1}|b(w)|dW$$

$$+ \, C_8 \int_{X/2}^{3X/2} e^{-W} \left[(X/(1+X))^{\nu} W^{-\nu} + X^{1-\nu} (1+X)^{\nu} W^{\nu-1} \right] X^{-1} |b(w)| dW$$

$$+ \, C_8 \int_{3X/2}^{\infty} e^{-W} \left[X^{\nu} (1+W)^{\nu} W^{1-\nu} + X^{1-\nu} W^{\nu} \right] X^{-2} |b(w)| dW,$$

where C_4 is independent of n.

To prove part i) of the lemma, we replace $|b(w)|$ by its bound of a constant and the result follows on estimating the integrals in (6.6.49).

To prove part ii) we observe that, with (6.4.27),

$$u(x) = \phi_1(x)(1 + C_2) + \int_0^{\infty} g(x,w) \left[w/(1 + w) \right]^{\nu} C(w) dw \tag{6.6.50}$$

$$+ \left[x/(1 + x) \right]^{\nu} C_3(x),$$

and use (6.6.49) with $|b(w)|$ replaced by $N^{-\nu} W^{\nu} C_1$, a bound for $\left[w/(1 + w) \right]^{\nu} C(w)$. ■

CHAPTER 7: CASE OF PREFERRED SET S_2

The next situation investigated is the case where S consists of several disjoint arcs, and the representative example studied in this chapter is that of two arcs, with preferred set S_2 of (3.1.6). The results of this chapter also follow from the work some years ago of Nuttall and Singh [24], who used an extended version of the Bernstein-Szegö integral equation. The significance of the present work is that it demonstrates that the results may also be derived from the integral equation of Chapter 5, and the derivation is more straightforward and elegant than the previous method.

7.1 Assumptions and Notation

This chapter concerns a set of the form S_2 (which we shall call S for the rest of the chapter) given in (3.1.6). We define

$$X(z) = \prod_{j=1}^{4} (z-b_j), \tag{7.1.1}$$

and

$$\phi(z) = \int_{b_1}^{z} (t-a)X(t)^{-1/2}dt. \tag{7.1.2}$$

The quantity a is obtained by solving (2.3.15), and $S = \left\{z\in\mathbf{C}:\mathrm{Re}\,\phi(z)=0\right\}$ consists (by assumption) of two disjoint analytic arcs joining b_1, b_2

and b_3, b_4. In $\mathbf{C}\backslash S$, the function $\mathrm{Re}\phi(z)$ is single-valued, and we choose its branch so that $\mathrm{Re}\phi(z)<0$, $z\in\mathbf{C}\backslash S$. We make use of the function $y(z)$, single-valued for $z\in\mathbf{C}\backslash S$, defined so that

$$(7.1.3) \qquad y(z) = X(z)^{1/2}, \qquad z\in\mathbf{C}\backslash S,$$

and

$$(7.1.4) \qquad y(z) \sim z^2, \qquad z \to \infty.$$

We later use arcs L_1, L_2 joining b_1 to b_2, b_3 and not intersecting S except at b_1.

We choose $\delta > 0$ sufficiently small so that the locus

$$(7.1.5) \qquad \Gamma_1 = \left\{z\in\mathbf{C}:\ \mathrm{Re}\phi(z) = -\delta\right\}$$

consists of two non-intersecting closed curves, each containing an arc of S. Similarly, we define

$$(7.1.6) \qquad \Gamma = \left\{z\in\mathbf{C}:\ \mathrm{Re}\phi(z) = -2\delta/3\right\},$$

and

$$(7.1.7) \qquad \Gamma_2 = \left\{z\in\mathbf{C}:\ \mathrm{Re}\phi(z) = -\delta/3\right\}.$$

The component of Γ containing a particular arc of S lies inside the corresponding component of Γ_1, and similarly for Γ_2, Γ.

The function f(z) to be approximated will have the form

$$f(z) = f_0 + (2\pi i)^{-1} \int_S y_+(t)^{-1} \sigma(t)(t-z)^{-1} dt, \tag{7.1.8}$$

where $\sigma(t)$ is analytic and non-zero for all t on or inside the two components of Γ_1.

Symbols may be used in this chapter with a meaning different from that of Chapter 6.

7.2 The Function h(z)

We are concerned with the case discussed in Chapter 5, Section 5.2. From (5.2.2) and (5.1.7) (with L replaced by S), we have

$$h_+(z)h_-(z) = \sigma(z)(z-c_n), \qquad z \in S. \tag{7.2.1}$$

The form of h(z) that satisfies (7.2.1) may be found from Lemma 5.2 of [24]. It depends on whether c_n is a zero of h(z) or not. We omit the special cases when $c_n = \infty$ or $c_n \in S$, which may be treated by modifying the arguments below. We have

Lemma 7.1. *The form of* h(z) *is one of the following, depending on whether* c_n*, and integers* η_1, η_2 *may be found to satisfy* (7.2.3), *or* (7.2.5) *below.*

i) $h(c_n) = 0$

$$(7.2.2) \qquad h(z) = \exp\Big\{y(z)(2\pi i)^{-1}\int_S (t-z)^{-1}y_+(t)^{-1}$$

$$\cdot\Big[\log\sigma(t) - \log(t-c_n) + 2n\log(t-b_1)\Big]dt$$

$$+\log(z-c_n) - n\log(z-b_1) - y(z)\sum_{j=1}^{2}\eta_j\int_{L_j}(t-z)^{-1}y(t)^{-1}dt\Big\},$$

with

$$(7.2.3) \qquad \int_S y_+(t)^{-1}\log\sigma(t)dt + i\pi\int_\infty^{c_n}y(t)^{-1}dt - 2\pi in\int_\infty^{b_1}y(t)^{-1}dt$$

$$-2\pi i\sum_{j=1}^{2}\eta_j\int_{L_j}y(t)^{-1}dt = 0.$$

ii) $h(c_n) \neq 0$

$$(7.2.4) \qquad h(z) = \exp\Big\{y(z)(2\pi i)^{-1}\int_S (t-z)^{-1}y_+(t)^{-1}$$

$$\cdot\Big[\log\sigma(t) + \log(t-c_n) + 2(n\;1)\log(t-b_1)\Big]dt$$

$$-(n-1)\log(z-b_1) - y(z)\sum_{j=1}^{2}\eta_j\int_{L_j}(t-z)^{-1}y(t)^{-1}dt\Big\},$$

with

$$\text{(7.2.5)}\qquad \int_S y_+(t)^{-1}\log\sigma(t)dt - i\pi\int_{\infty}^{c_n}y(t)^{-1}dt - 2\pi i(n-1)\int_{\infty}^{b_1}y(t)^{-1}dt$$

$$-2\pi i\sum_{j=1}^{2}\eta_j\int_{L_j}y(t)^{-1}dt = 0.$$

Proof. The expressions given for h(z) are analytic for $z\in\mathbf{C}\backslash S$, since the discontinuity of the exponent at a point $z\in L_j$ is an integer multiple of $2\pi i$. The Plemelj formula [20] shows that (7.2.1) is satisfied. The dominant behaviour of the exponent in case i) near $z=\infty$ is $-(n-1)\log z$ provided that the coefficient of the term of order z coming from the integrals is zero. This requires

$$\text{(7.2.6)}\qquad (2\pi i)^{-1}\int_S y_+(t)^{-1}\Big[\log\sigma(t) - \log(t-c_n) + 2n\log(t-b_1)\Big]dt$$

$$-\sum_{j=1}^{2}\eta_j\int_{L_j}y(t)^{-1}dt = 0.$$

As in Lemma 5.2 of [24], this may be rewritten as (7.2.3). The proof in case ii) is similar. ∎

Equations (7.2.3) and (7.2.5) may be written as a single equation if c_n is regarded as a point on the Riemann surface

$$(7.2.7) \qquad \mathbb{R}\colon\ y^2 = X(z),$$

on sheet 1 in case i) and sheet 2 in case ii). This equation has the form

$$(7.2.8) \qquad \int_{\infty^{(1)}}^{c_n} y(t)^{-1}dt = n\int_{\infty^{(1)}}^{\infty^{(2)}} y(t)^{-1}dt + 2\sum_{j=1}^{2}\eta_j\int_{L_j} y(t)^{-1}dt$$

$$-(i\pi)^{-1}\int_{S} y_+(t)^{-1}\log\sigma(t)dt,$$

where the paths of integration are chosen appropriately on $\mathbb{R}$. [24]. The theory of elliptic functions [39] shows that (7.2.8) always has a unique solution. If c_n is on sheet 1 then we have case i) and if on sheet 2, case ii). Although c_n can be given explicitly in terms of elliptic functions, we shall make no use of this information, and give asymptotic results that are valid wherever c_n is located.

In the analysis of the behaviour of h(z) for large n, we need the real quantities B_1, B_2 defined uniquely by

$$(7.2.9) \qquad \sum_{j=1}^{2} B_j\int_{L_j} y(t)^{-1}dt = \int_{\infty}^{b_1} y(t)^{-1}dt.$$

In Lemma 6.7 of [24], the equivalent of the following lemma was proved.

Lemma 7.2. *If $\phi(z)$ is made single-valued in $\mathbb{C}\backslash S$ by the addition of cuts along L_1, L_2, then one evaluation of $\phi(z)$ may be written as*

$$\phi(z) = \phi^{*} + y(z)\left[(\pi i)^{-1}\int_S (t-z)^{-1}y_{+}(t)^{-1}\log(t-b_1)dt \right.$$

$$\left. + \sum_{j=1}^{2} B_j \int_{L_j} y(t)^{-1}(t-z)^{-1}dt\right] - \log(z-b_1), \tag{7.2.10}$$

where ϕ^{*} is a pure imaginary.

In Lemma 4.1 of [24], it was shown that the quantities η_j, $j = 1, 2$ satisfying (7.2.8) may be written as

$$\eta_j = -nB_j + \xi_j, \qquad j = 1, 2, \tag{7.2.11}$$

where $|\xi_j|$ are bounded by a constant independent of n.

7.3 Analysis of Integral Equation

As before we begin by changing the contour in the integral equations for U(z) found in Section 5.2. Suppose n is such that case i) of Lemma 7.1 holds and U(z) has a pole at $z=c_n$. Using (7.2.1), we rewrite (5.2.5) as

$$(7.3.1) \qquad U(z) = 1 - \int_S \Big[K_+(z,t)h_+(t)^2U_+(t) - K_-(z,t)h_-(t)^2U_-(t)\Big] \cdot \Big[\sigma(t)(t-c_n)\Big]^{-1} dt,$$

where

$$(7.3.2) \quad K(z,t)=(4\pi i)^{-1}\Big[-(y(t)-y(z))(t-z)^{-1}-(y(c_n)+y(z))(c_n-z)^{-1}\Big]y(t)^{-1}$$

We note that, since $h(c_n)=0$, the function $K(z,t)h(t)^2U(t)(\sigma(t)(t-c_n))^{-1}$ is analytic for t inside Γ_1, whatever the location of c_n, so that (7.3.1) is equivalent to

$$(7.3.3) \qquad U(z) = 1 + \int_{\Gamma_1} K(z,t)h(t)^2\Big[\sigma(t)(t-c_n)\Big]^{-1}U(t)dt.$$

The contour Γ_2 could equally well have been used.

A similar analysis shows that in case ii)

$$(7.3.4) \qquad U(z) = 1 + \int_{\Gamma_1} K'(z,t)h(t)^2\Big[\sigma(t)(t-c_n)\Big]^{-1}U(t)dt,$$

where

$$(7.3.5) \quad K'(z,t)=(4\pi i)^{-1}\Big[-(y(t)-y(z))(t-z)^{-1}+(y(c_n)-y(z))(c_n-z)^{-1}\Big]y(t)^{-1}$$

With these preliminaries, we are able to prove the main result of this chapter.

Theorem 7.3 *Provided that* n *is not such that* c_n *is near* ∞, *we have uniformly for all* z, *as* $n \to \infty$,

(7.3.6) $$U(z) = 1 + \exp(-2n\delta/3)O(1)$$

$$+ \exp(-2n\delta/3)O(1)\begin{cases}(y(c_n)+y(z))(c_n-z)^{-1} & \textit{case i),}\\ (y(c_n)-y(z))(c_n-z)^{-1} & \textit{case ii).}\end{cases}$$

The result includes the limits of $U(z)$ as $z \to S$ from either side.

Proof. If n is such that c_n lies inside Γ, we use the contour Γ_1 in (7.3.3) or (7.3.4). Otherwise we use the contour Γ_2. We write for $t\in\Gamma_1$ or Γ_2,

(7.3.7) $$h(t)=\exp\left[n\phi(t)\right]\psi(t)\theta(t)G(t)$$

where

(7.3.8) $$\psi(t)=\exp\left[y(t)(2\pi i)^{-1}\int_S y_+(u)^{-1}\log\sigma(u)(u-t)^{-1}du\right],$$

(7.3.9) $$\theta(t)=\exp\left[-y(t)\sum_{j=1}^{2}\xi_j\int_{L_j} y(u)^{-1}(u-t)^{-1}du\right]$$

For case i), we have,

$$(7.3.10) \qquad G(t) = (t-c_n)\exp\left[-y(t)(2\pi i)^{-1}\int_S y_+(u)^{-1}\log(u-c_n)(u-t)^{-1}du\right],$$

and for case ii), we have,

$$(7.3.11) \quad G(t) = \exp\left\{y(t)(2\pi i)^{-1}\int_S y_+(u)^{-1}(u-t)^{-1}\left[\log(u-c_n)-2\log(u-b_1)\right]du + \log(t-b_1)\right\}.$$

Under the conditions stated on c_n, t, the exponents of $\psi(t)$, $\theta(t)$, G(t) are uniformly bounded so that

$$(7.3.12) \qquad |h(t)| = \exp\left[nRe\phi(t)\right]O(1).$$

Thus, in every case,

$$(7.3.13) \qquad h(t)^2\left[\sigma(t)(t-c_n)\right]^{-1} = \exp(-2n\delta/3)O(1), \qquad n\to\infty,$$

uniformly for $t\in\Gamma_1$ or Γ_2 as required. The result follows. ∎

7.4 Asymptotic Formulas for Pade Polynomials

In this case we define

$$\chi_2(z) = \chi(z) = (z-c_n)h(z)^{-1}$$

(7.4.1) $$\chi_1(z) = -f(z)\chi_2(z)$$

$$R_0(z) = h(z)y(z)^{-1}$$

where y(z), f(z) and h(z) are given respectively by (7.1.3), (7.1.8) and (7.2.2) or (7.2.4). These functions satisfy (3.3.3), (3.3.4) as predicted.

We now have

(7.4.2) $$p_2(z) = Q(z)\chi(z)$$

where Q(z) is given by (5.2.9) or (5.2.10). As in Section 7.3, we may rewrite these equations as

(7.4.3) $$Q(z) = 1 + \int_{\Gamma_1} \tilde{K}(z,t)h(t)^2\left[\sigma(t)(t-c_n)\right]^{-1}U(t)dt,$$

if $h(c_n)=0$,

(7.4.4) $$Q(z) = 1 + \int_{\Gamma_1} \tilde{K}'(z,t)h(t)^2\left[\sigma(t)(t-c_n)\right]^{-1}U(t)dt,$$

if $h(c_n)\neq 0$, where

(7.4.5) $$\tilde{K}(z,t)=(4\pi i)^{-1}\left[-(y(t)+y(z))(t-z)^{-1}-(y(c_n)-y(z))(c_n-z)^{-1}\right]y(t)$$;

$$(7.4.6)\quad \tilde{K}'(z,t)=(4\pi i)^{-1}\left[-(y(t)+y(z))(t-z)^{-1}+(y(c_n)+y(z))(c_n-z)^{-1}\right]y(t)^{-1}$$

With the current definitions, (6.5.2) and (6.5.3) are still correct.

The proof of Theorem 4.2 i) follows on substituting the estimates of Theorem 7.3 into the formulas above for $Q(z)$, which gives $p_2(z)$ through (7.4.2) and $p_1(z)$ from (6.5.5). Part ii) is obtained immediately on using (6.5.2), (6.5.3).

CHAPTER 8: CASE OF PREFERRED SET S_3

For many functions $f(z)$ with branch points, the preferred set S will have one or more points at which several arcs of S intersect. The simplest example of this situation is the set S_3 described in Section 3.1. Three arcs of S end at the point a. The points b_1, b_2, b_3 are definitely branch points of $f(z)$, but the point a will in general, not be a branch point of $f(z)$. An example of a function having a preferred set of the form S_3 is given in Section 2.4.

The method of Nuttall and Singh [24] does not appear to apply to situations such as the preferred set S_3. The purpose of this chapter is to show how to use the integral equation of Section 5.2 to obtain the asymptotic behavior of the Padé polynomials for certain functions having a preferred set of the form S_3. No doubt the techniques described here can be generalized to other types of sets S for which several arcs intersect.

8.1 Notation and Assumptions

Given distinct points $b_j \in \mathbf{C}$, j=1, 2, 3, it has been shown [27] that the point a satisfying (2.3.15)

$$\mathrm{Re}\phi(b_2) = \mathrm{Re}\phi(b_3) = 0$$

is unique, where $\phi(z)$ is given by (3.1.7). The set

$$S = S_3 = \left\{z\in\mathbf{C} : \mathrm{Re}\,\phi(z) = 0\right\} \tag{8.1.1}$$

consists of three analytic arcs s_1, s_2, s_3 joining the point a to b_1, b_2, b_3. We assume that b_1, b_2, b_3 occur in counterclockwise order around a. In $\mathbf{C}\setminus S$, $\mathrm{Re}\,\phi(z)$ is single-valued and we choose that $\mathrm{Re}\,\phi(z)<0$.

Suppose that $\delta>0$. We define closed curves Γ, Γ_1, Γ_2 as in (7.1.5), (7.1.6) and (7.1.7). In this case, each curve contains a single component. They are nested as shown in Figure 8.1. The function $\mathrm{Im}\,\phi(z)$ is not single-valued, but if we choose $\mathrm{Im}\,\phi(z)$ to be continuous along a locus $\mathrm{Im}\,\phi(z)=\text{const.}$, such a locus is an arc orthogonal to the curves $\mathrm{Re}\,\phi(z)=\text{const.}$ and is independent of the particular evaluation of $\mathrm{Im}\,\phi(z)$. We define Ω_j, $j=1, 2, 3$ as the locus $\mathrm{Im}\,\phi(z)=\text{const.}$ running from b_j to Γ_2. In the sector bounded by s_2, s_3 there is an arc

$$\Lambda_1 = \left\{z\in\mathbf{C} : \mathrm{Im}\,\phi(z)=\text{const.}\right\} \tag{8.1.2}$$

running from a to Γ_2, and similarly for Λ_2, Λ_3. That part of Γ_2 between the intersections with Ω_1, Λ_3 will be denoted by γ_{23} and similarly for other portions as in Figure 8.1. The region bounded by s_1, Ω_1, γ_{23}, Λ_3 will be called Σ_{23} and so on.

We define $y(z)$ by

$$y^2(z) = (z-a)\prod_{j=1}^{3}(z-b_j) \tag{8.1.3}$$

and choose that $y(z)$ be single-valued for $z\in\mathbf{C}\setminus S$, with $y(z)\sim z^2$ as $z\to\infty$.

Without loss of generality, we assume that $a=0$, and that the tangent at point a to s_1 lies along the positive real axis. We assume that $z^{1/2}$ is made single-valued in the interior of Γ_2 by placing a cut along s_1, Ω_1 and that near a, $z_+^{1/2}$ is real and positive. Suppose that $D = (-b_1 b_2 b_3)^{1/2}$ is such that near $z=0$, in the sector 1-2,

$$\text{sector 1-2:} \qquad y(z) \approx iDz^{1/2}. \tag{8.1.4}$$

Then in the other sectors we have

$$\begin{aligned} &\text{sector 2-3:} \qquad y(z) \approx -iDz^{1/2}, \\ &\text{sector 3-1:} \qquad y(z) \approx iDz^{1/2}. \end{aligned} \tag{8.1.5}$$

Near $z=0$, in the 1-2 sector, we have an evaluation of $\phi_3(z)$ of $\phi(z)$ given by

$$\begin{aligned} \phi_3(z) &= -\int_0^z ty(t)^{-1}dt \\ &\approx iD^{-1}\int_0^z t^{1/2}dt \\ &= 2i(3D)^{-1}z^{3/2} \\ &= 2r^{3/2}(3D)^{-1}\left[-\sin(3\theta/2) + i\cos(3\theta/2)\right], \end{aligned} \tag{8.1.6}$$

if $z = re^{i\theta}$, r, θ real.

In order that $\mathrm{Re}\phi_3(r) = 0$ as assumed by our choice of direction of s_1, it is necessary that D be real and positive. It follows from this analysis that the tangents at 0 to s_2 and s_3 make angles $2\pi/3$ and $4\pi/3$, respectively, with s_1. Also the directions of the tangents at 0 to Λ_j are θ_j, where

(8.1.7)
$$\theta_1 = \pi$$
$$\theta_2 = 5\pi/3$$
$$\theta_3 = \pi/3.$$

We define evaluations $\phi_1(z)$, $\phi_2(z)$ similarly to the above.

After describing the properties of S and $\phi(z)$, we now come to the assumptions about $f(z)$. We assume

(8.1.8)
$$f(z) = f_0 + (2\pi i)^{-1} \sum_{j=1}^{3} \int_{s_j} \omega_j(t)(t-z)^{-1}dt.$$

We assume that $\omega_1(z)$ has an non-zero analytic continuation from s_1 into $\Sigma_{23} \cup \Sigma_{32}$ such that $(z-b_1)^{1/2}\omega_1(z)$ is analytic in $s_1 \cup \Omega_1 \cup \Sigma_{23} \cup \Sigma_{32}$ and also in a neighbourhood of point a, and is non-zero at b_1, and similarly for $\omega_2(z)$, $\omega_3(z)$. We also assume that, in a given sector, the limit of $f(z)$ as $z \to 0$ is independent of direction. This leads to the conclusion that

(8.1.9)
$$\sum_{j=1}^{3} \Delta_j f(z) = 0, \qquad \text{as } z \to 0,$$

where $\Delta_j f(z)=f_+(z)-f_-(z)$ taken across s_j. The Plemelj formula gives

$$\Delta_j f(z) = \omega_j(z) \tag{8.1.10}$$

so that

$$\sum_{j=1}^{3} \omega_j(0) = 0. \tag{8.1.11}$$

8.2 Distorted Integral Equation

We begin with the integral equation of Section 5.2, which, for preferred set S_3, has exactly the same appearance as that used in Chapter 7, provided that we now define $y(z)$ by (8.1.3).

The formulas for $h(z)$ are given by (7.2.2), (7.2.4), with the understanding that $y(z)$ above is used in the definition of $\sigma(z)$,

$$\sigma(z) = y_+(z)\Big[f_+(z) - f_-(z)\Big], \qquad z \in S. \tag{8.2.1}$$

In this chapter, our concern is the complications introduced into the analysis by the arcs meeting at $z=a$. The effect of the zero c_n will be as in Chapter 7, except when c_n is close to zero. To save writing we assume that $h(c_n)=0$ and that c_n is outside Γ. The other possibilities can be handled as in Chapter 7.

As in (7.3.1) the integral equation for $U(z)$ may be written

$$(8.2.2)\quad U(z) = C_0/2 + \sum_{j=1}^{3} \int_{s_j} \left[K_+(z,t)h_+(t)^2U_+(t) - K_-(z,t)h_-(t)^2U_-(t)\right] \cdot \left[\sigma(t)(t-c_n)\right]^{-1} dt,$$

where $K(z,t)$ is given by (7.3.2). Since

$$(8.2.3)\qquad K(z,t)h(t)^2\left[\sigma(t)(t-c_n)\right]^{-1}U(t)$$

is analytic in t for t in the region Σ_{23} bounded by the curves s_1, Ω_1, γ_{23}, Λ_3, we may rewrite

$$(8.2.4)\qquad \int_{s_1} K_+(z,t)h_+(t)^2U_+(t)\left[\sigma(t)(t-c_n)\right]^{-1} dt$$

$$= \int_{\Omega_1 \cup \gamma_{23} \cup \Lambda_3} K(z,t)h(t)^2U(t)\left[\sigma(t)(t-c_n)\right]^{-1} dt,$$

and similarly for the other parts of (8.2.2). Because of the assumption in Section 8.1 about $\omega_j(z)$, the contributions from Ω_j, j=1, 2, 3, will cancel, and (8.2.2) becomes

$$(8.2.5)\qquad U(z) = C_0/2 + \int_{\Gamma_2 \bigcup_j \Lambda_j} K(z,t)H(t)U(t)dt, \qquad z \in \Gamma_2 \bigcup_j \Lambda_j,$$

where

$$(8.2.6)\qquad H(t) = -h(t)^2\left[\sigma_j(t)(t-c_n)\right]^{-1}, \qquad t\in\Gamma_2$$

$$= (t-c_n)^{-1}h(t)^2\left[\sigma_2(t)^{-1} - \sigma_3(t)^{-1}\right], \qquad t\in\Lambda_1,$$

etc.

By $\sigma_j(t)$ we mean the analytic continuation of $\sigma(t)$ as defined on S_1.

The behaviour of $H(t)$ near $t=a$ is given by

Lemma 8.1 *With the assumptions given above,*

$$(8.2.7)\quad H(t) = -i\exp\left[2n\phi_j(z)\right]\left[1 + E(|t-a|) + O(|t-a|)\right], \quad t\in\Lambda_j,\ j=1,2,3.$$

where

$$(8.2.8)\qquad E(s) = O(|s^{1/2}\log s|), \qquad s\to 0.$$

Proof. We use the same techniques as for the proof of Lemma 6.3. It is important to remember (8.1.11). ∎

8.3 Analysis of Integral Equation

The integral equation (8.2.5) may be analyzed in a manner similar to that used for (6.2.6). For large n, the kernel of the equation is very small for all t on the contour of integration except for values of

t near a. Such values can now occur on the three arcs Λ_j, $j=1, 2, 3$, whereas, in Chapter 6, t had to be on the single arc Γ_2 if t were near to the exceptional point $t=1$.

To obtain the framework needed for a rigorous discussion, we now use the same method that was used in Chapter 6, and find an approximation to the kernel of (8.2.5) valid when t and z are near the point a. We approximate the arcs Λ_j, $j=1, 2, 3$ by their tangents at $z=a$, obtaining

$$\Lambda_j: t = \exp(i\theta_j)r, \qquad j=1, 2, 3, \qquad r \text{ real}, r\geq 0. \tag{8.3.1}$$

From (8.3.1), we thus have approximations to $\phi_j(t)$, $t \in \Lambda_j$, $j=1, 2, 3$, given by

$$\phi_j(t) = (-2/3)D^{-1}r^{3/2}, \qquad j=1, 2, 3. \tag{8.3.2}$$

For t, z near to the point a, the most important part of the kernel $k(z,t)$ used in (8.2.5) is

$$\Big[y(t) - y(z)\Big]y(z)^{-1}(t-z)^{-1} \tag{8.3.3}$$

which becomes

$$\Big[t^{1/2}(t^{1/2} + z^{1/2})\Big]^{-1} \text{ or } \Big[t^{1/2}(t^{1/2} - z^{1/2})\Big]^{-1} \tag{8.3.4}$$

when we use the approximations (8.1.4) and (8.1.5)

$$y(z) \approx \pm iDz^{1/2}, \qquad y(t) \approx iDt^{1/2},$$

depending on whether the signs for y(z), y(t) are the same or different. The approximate integral equation is rewritten most conveniently by setting

$$\text{(8.3.5)} \qquad \begin{aligned} z &= \exp(i\theta_j)\zeta^2, \qquad z\in\Lambda_j \\ t &= \exp(i\theta_j)\tau^2, \qquad t\in\Lambda_j \end{aligned} \qquad j=1,\ 2,\ 3,$$

and defining

$$\text{(8.3.6)} \qquad U_j(\zeta) = U(z), \qquad z\in\Lambda_j, \qquad j=1,\ 2,\ 3.$$

As a result, (8.2.5) becomes

(8.3.7)

$$U_1(\zeta) = C_0/2 - (2\pi)^{-1}\int_0^\infty e^{-N\tau^3}\left[(\tau+\zeta)^{-1}U_1(\tau)+(\tau+\omega\zeta)^{-1}U_2(\tau)+(\tau+\bar{\omega}\zeta)^{-1}U_3(\tau)\right]d\tau$$

$$U_2(\zeta) = C_0/2 - (2\pi)^{-1}\int_0^\infty e^{-N\tau^3}\left[(\tau+\bar{\omega}\zeta)^{-1}U_1(\tau)+(\tau+\zeta)^{-1}U_2(\tau)+(\tau+\omega\zeta)^{-1}U_3(\tau)\right]d\tau$$

$$U_3(\zeta) = C_0/2 - (2\pi)^{-1}\int_0^\infty e^{-N\tau^3}\left[(\tau+\omega\zeta)^{-1}U_1(\tau)+(\tau+\bar{\omega}\zeta)^{-1}U_2(\tau)+(\tau+\zeta)^{-1}U_3(\tau)\right]d\tau$$

where $\omega = \exp(i2\pi/3)$, $\bar{\omega} = \exp(-i2\pi/3)$, and $N = 4n/(3D)$. As in the analogous case in Chapter 6, (6.4.3), the upper limit of integration has been taken to infinity.

The procedure for analyzing this case follows closely the pattern of Chapter 6. We will need the solution of (8.3.7) when $C_0/2$ is replaced in each equation by an arbitrary inhomogeneous term. The main result required follows from

Lemma 8.2 *If $J_j(\zeta)$ are given functions, then the solution of*

(8.3.8)

$$W_1(\zeta) = J_1(\zeta) - (2\pi)^{-1}\int_0^\infty e^{-N\tau^3}\Big[(\tau+\zeta)^{-1}W_1(\tau) + (\tau+\omega\zeta)^{-1}W_2(\tau) + (\tau+\bar{\omega}\zeta)^{-1}W_3(\tau)\Big]d\tau$$

$$W_2(\zeta) = J_2(\zeta) - (2\pi)^{-1}\int_0^\infty e^{-N\tau^3}\Big[(\tau+\bar{\omega}\zeta)^{-1}W_1(\tau) + (\tau+\zeta)^{-1}W_2(\tau) + (\tau+\omega\zeta)^{-1}W_3(\tau)\Big]d\tau$$

$$W_3(\zeta) = J_3(\zeta) - (2\pi)^{-1}\int_0^\infty e^{-N\tau^3}\Big[(\tau+\omega\zeta)^{-1}W_1(\tau) + (\tau+\bar{\omega}\zeta)^{-1}W_2(\tau) + (\tau+\zeta)^{-1}W_3(\tau)\Big]d\tau$$

where $0 \le \zeta < \infty$, is given by

$$W_1(\zeta) = \Big[v_1(x) + \zeta v_2(x) + \zeta^2 v_3(x)\Big]/3,$$

(8.3.9)
$$W_2(\zeta) = \Big[v_1(x) + \bar{\omega}\zeta v_2(x) + \omega\zeta^2 v_3(x)\Big]/3, \qquad 0 \le x < \infty$$

$$W_3(\zeta) = \Big[v_1(x) + \omega\zeta v_2(x) + \bar{\omega}\zeta^2 v_3(x)\Big]/3,$$

with $x=\zeta^3$, *where* $v_j(x)$, $j=1, 2, 3$, *satisfy*

$$v_1(x) = j_1(x) + \int_0^\infty V(x,w)v_1(w)\,dw,$$

(8.3.10)
$$v_2(x) = \zeta j_2(x) - \int_0^\infty V(x,w)v_2(w)\,dw, \qquad 0\le x<\infty$$

$$v_3(x) = \zeta^2 j_3(x) - \int_0^\infty V(x,w)v_3(w)\,dw.$$

We have used

$$j_1(x) = J_1(\zeta) + J_2(\zeta) + J_3(\zeta),$$

(8.3.11)
$$j_2(x) = J_1(\zeta) + \omega J_2(\zeta) + \bar{\omega} J_3(\zeta), \qquad 0\le x<\infty$$

$$j_3(x) = J_1(\zeta) + \bar{\omega} J_2(\zeta) + \omega J_3(\zeta),$$

and

(8.3.12)
$$V(x,w) = -\lambda_{1/6} e^{-Nw}(x+w)^{-1},$$

where

(8.3.13)
$$\lambda_{1/6} = (\sin\pi/6)/\pi = (2\pi)^{-1}$$

as in (6.4.4).

Proof. We write

$$
\begin{aligned}
v_1(x) &= W_1(\zeta) + W_2(\zeta) + W_3(\zeta), \\
\zeta v_2(x) &= W_1(\zeta) + \omega W_2(\zeta) + \bar{\omega} W_3(\zeta), \qquad 0 \le x < \infty \\
\zeta^2 v_3(x) &= W_1(\zeta) + \bar{\omega} W_2(\zeta) + \omega W_3(\zeta).
\end{aligned}
\tag{8.3.14}
$$

Substitution of the right-hand sides of (8.3.8) into (8.3.14), some algebraic manipulation, and change of variables

$$
\begin{aligned}
x &= \zeta^3, \\
w &= \tau^3,
\end{aligned}
\tag{8.3.15}
$$

gives the result. ∎

Lemma 6.9 gives the solution of the equations (8.3.10). When applied to (8.3.7), we find that

$$
\begin{aligned}
j_2(x) &= j_3(x) = 0, \\
j_1(x) &= 3C_0/2,
\end{aligned}
\tag{8.3.16}
$$

so that, with $C_0/2 = 1$, we find that the solution of (8.3.7) is, with $x = \zeta^3$, (8.3.17) $$U_j(\zeta) = \pi^{-1/2}N^{1/2}x^{1/2}e^{Nx/2}K_{1/3}(Nx/2), \quad j=1,\ 2,\ 3.$$

The rigorous analysis of this case now proceeds in a manner similar to that followed in Chapter 6. We will describe the results of the various steps needed and omit details of proofs where there is no significant difference from the arguments of Chapter 6.

We begin by setting up a Banach space B_3 in which to analyze (8.2.5). We define a mapping

(8.3.18) $$\rho_2 : \Lambda_1 \to \Lambda_2$$

by letting

(8.3.19) $$z_2 = \rho_2(z_1), \qquad z_2 \in \Lambda_2, \quad z_1 \in \Lambda_1,$$

if

(8.3.20) $$\mathrm{Re}\phi(z_2) = \mathrm{Re}\phi(z_1),$$

and similarly $\rho_3 : \Lambda_1 \to \Lambda_3$. For a function $U(z)$ defined on $\Gamma_2 \cup \Lambda_1$ we define $\|U\|_3$ by

$$(8.3.21)\qquad \|U\|_3 = \mathrm{Max}\begin{cases} \sup\limits_{z\in\Lambda_1}\left|\left[U(z) + U(\rho_2(z)) + U(\rho_3(z))\right]x^{\mu}\right|, \\[2ex] \sup\limits_{z\in\Lambda_1}\left|\left[U(z) + \omega U(\rho_2(z)) + \bar{\omega}U(\rho_3(z))\right]x^{\mu-1/3}\right|, \\[2ex] \sup\limits_{z\in\Lambda_1}\left|\left[U(z) + \bar{\omega}U(\rho_2(z)) + \omega U(\rho_3(z))\right]x^{\mu-2/3}\right|, \\[2ex] \sup\limits_{z\in\Gamma_2}|U(z)|, \end{cases}$$

where $1/2<\mu<2/3$ and $z = -x^{2/3}$. The space $\mathbb{B}_3$ consists of functions $U(z)$, continuous except at $z = 0$, for which $\|U\|_3<\infty$.

In terms of the linear operator $\mathbb{K}_3$ with kernel $K(z,t)\mathbb{H}(t)$, (8.2.5) may be written as

$$(8.3.22)\qquad U = C_0/2 + \mathbb{K}_3U.$$

Corresponding to Lemma 6.5, we have

Lemma 8.3 *The operator* $\mathbb{K}_3: \mathbb{B}_3 \to \mathbb{B}_3$ *is bounded, and there exists a value* n_0 *and a quantity* $k_3<1$ *such that, for all* $n>n_0$, $\|\mathbb{K}_3\|_3 < k_3$.

Proof. The method is to take any $U\in\mathbb{B}_3$ with $\|U\|_3=1$ and define $U'=\mathbb{K}_3U$. We first show that

(8.3.23) $$\sup_{z\in\Gamma_2} |U'(z)| \to 0 \qquad \text{as } n\to\infty.$$

The dominant contribution to

(8.3.24) $$U'(z) = \int_{\Gamma_2 \cup_j \Lambda_j} K(z,t)\mathbb{H}(t)U(t)dt,$$

comes from values of t near t=0. We have for $z\in\Gamma_2$, $t\approx 0$,

(8.3.25) $$K(z,t)\mathbb{H}(t)dt \approx (2\pi iD)^{-1}\left[y(z)/z + (y(c_n) + y(z))(c_n - z)^{-1}\right]$$

$$\cdot \exp\left[-N\tau^3 + i\pi(\theta_j/2 + \psi_j)\right]d\tau,$$

with

(8.3.26) $$t = \exp(i\theta_j)\tau^2, \qquad \tau>0,$$

corresponding to $t\in\Lambda_j$, j=1, 2, 3, and $\psi_1=1$, $\psi_2=\psi_3=0$. This gives

(8.3.27) $$\mathbb{K}_3(z,t)dt = F(z)\exp(-N\tau^3)d\tau \begin{bmatrix} 1 & & j = 1 \\ \bar{\omega} & & j = 2 \\ \omega & & j = 3 \end{bmatrix},$$

where F(z) is a bounded function of z, $z \in \Gamma_2$. The contribution to $U'(z)$ becomes

$$(8.3.28)\qquad U'(z) = F(z)\int_0^{\tau_0} e^{-N\tau^3}\left[U(t) + \bar{\omega}U(\rho_2(t)) + \omega U(\rho_3(t))\right]d\tau,$$

with

$$(8.3.29)\qquad t = \exp(i\theta_1)\tau^2.$$

The term in brackets is bounded by $\tau^{2-3\mu}$ and the result follows.

To show that the remaining contributions to $\|U'\|_3$ are $< k_3$, we need to consider $z\in\Lambda_j$, j=1, 2, 3, and values of z and of t in (8.3.24) near to zero are the most important. Thus we approximate as in (8.3.7) and use the results of Lemma 8.2 to express each of the three combinations of $U'(z)$, $U'(\rho_2(z))$, $U'(\rho_3(z))$, used in (8.3.21) in terms of an integral over the same combination of $U(z)$, $U(\rho_2(z))$, $U(\rho_3(z))$, and we have

$$(8.3.30)\quad U'(z)+U'(\rho_2(z))+U'(\rho_3(z))\approx\int_0^{x_0} V(x,t)\left[U(t)+U(\rho_2(t))+U(\rho_3(t))\right]dt,$$

where z, t $\in \Lambda_1$ and

$$(8.3.31)\qquad t = \exp(i\theta_1)w^{2/3},$$

and two similar equations. The proof of Lemma 6.7 shows that the contribution of the right-hand side of (8.3.30) to $\|U'\|_3$ satisfies the statement of Lemma 8.3.

The proof is completed in a straightforward manner by showing that the corrections to the above approximations approach zero as $n \to \infty$. The fact that the error term $E(|z-a|)$ in (8.2.7) is independent of j is important at several points in the analysis. ∎

As in Chapter 6, Lemma 8.3 implies that (8.3.22) has a unique solution for large n, and we may choose the Padé polynomial normalization so that $C_0/2=1$. The theorem below is analogous to Theorem 6.6 and is proved in the same way.

Theorem 8.4 *With the assumptions of Section* 8.1, *there exists a value* n_0, *such that, for all* $n > n_0$, *the remainder function* $R(z)$ *and the Padé polynomials* $p_j(z)$, $j=1, 2$, *are unique up to a constant factor. For any given* $\varepsilon > 0$, *when* $|z-a| > \varepsilon$ *and* n *is such that* $\varepsilon < |c_n - a| < \infty$, $n > n_0$, *the asymptotic form of* $R(z)$ *is given uniformly by*

$$(8.3.32)\quad R(z)=y(z)^{-1}h(z)\left[1+o(1)+o(1)\begin{cases}(y(c_n)+y(z))(c_n-z)^{-1} & \textit{case i)}\\[2ex] (y(c_n)-y(z))(c_n-z)^{-1} & \textit{case ii)}\end{cases}\right].$$

The form of $U(z)$ near $z=a$ may now be obtained. The proof follows the same pattern as that for Theorem 6.12. In terms of the function

$$(8.3.33)\qquad \phi_0(x) = \pi^{-1/2}N^{1/2}x^{1/2}e^{Nx/2}K_{1/3}(Nx/2)$$

the result is, with a=0,

Theorem 8.5 *Suppose that* $n>n_0$ *is such that* $|c_n| > \varepsilon_1$, *with* ε_1 *suitably small, fixed, and that* c_n *is not near* ∞. *With the assumptions of Section 8.1, the asymptotic form of* $U(z)$ *is given uniformly in the sector containing* Λ_j, $j=1, 2, 3$, *by*

$$U(z) = \phi_0\left[(z \exp(-i\theta_j))^{3/2}\right]\left[1 + o(1)\right], \qquad n \to \infty \tag{8.3.34}$$

for $|\arg(z-\theta_j)| \le 2\pi/3$ *and* $|z| \le \varepsilon_1$.

CHAPTER 9: CONCLUSIONS AND SUGGESTIONS

We have developed methods that lead to the asymptotic form of diagonal Padé polynomials for certain functions with branch points. We have shown how our techniques apply to cases for which the branch point is not of square root type, and for which the preferred set S contains three intersecting arcs. This is the first time that strong Padé polynomial asymptotics have been derived in such situations, except for certain special examples. We have also shown that our techniques apply in the case when S has more than one component arc, so that the polynomials may have a zero that is not close to S.

Our results suggest a number of possibilities for further research.

1) Combine the techniques to apply to cases when several branch points are not of square root type and arcs of S intersect each other.
2) Place weaker restriction on the discontinuity $\sigma(z)$ - e.g. that $\sigma(z)$ be smooth on S instead of analytic in a neighbourhood of S.
3) Extend the method to the case when more than three arcs of S intersect.
4) Analyze the special cases where c_n is near an exceptional point.
5) Study the case when $\sigma(z)$ has a zero on S.
6) Combine all the above to work out asymptotics for any algebraic function.
7) Apply the same ideas to Hermite-Padé approximants [27].

There seems to be no reason why progress could not be made on all these possibilities. The result would be a deeper understanding of the theory of Padé approximants and its generalizations.

APPENDIX 1

The aim of this appendix is to investigate the asymptotic behaviour of following two types of integral near the point $z = 0$. The integrals $I_1(z)$ and $I_2(z)$ are defined by

$$I_1(z) = \zeta(z) z^{1/2} \int_{-d}^{0} t_+^{-1/2} \zeta(t)^{-1} (t-z)^{-1} dt,$$

$$I_2(z) = \zeta(z) z^{1/2} \int_{-d}^{0} t_+^{-1/2} \zeta(t)^{-1} \log t (t-z)^{-1} dt,$$

where the function $\zeta(z)$ is analytic and non-zero near $z = 0$, the function $z^{1/2}$ is analytic in the complex plane cut along the negative real axis, and we assume that $d>0$. Then, the asymptotic forms of $I_1(z)$ and $I_2(z)$ near $z = 0$ are given by

(A1.1) $$I_1(z) = i\pi + O(|z^{1/2}|), \qquad z\to 0,$$

(A1.2) $$I_2(z) = i\pi \log z - \pi^2 + O(|z^{1/2} \log z|), \qquad z\to 0.$$

Proof. Let

(A1.3) $$z = x^2, \qquad t = -y^2,$$

$$\zeta(z) = \zeta(0)\left[1 + O(z)\right], \tag{A1.4}$$

it follows that

$$I_1(z) = \zeta(0)\left[1 + O(x^2)\right]x\int_{y_0}^{0}(iy)^{-1}\zeta(0)^{-1}\left[1 + O(y^2)\right](x^2+y^2)^{-1}2y\,dy, \tag{A1.5}$$

where $y_0 = d^{1/2}$, and this leads to

$$I_1(z) = \left[1 + O(x^2)\right]2ix\int_{0}^{y_0}\left[1 + O(y^2)\right](x^2+y^2)^{-1}dy. \tag{A1.6}$$

Now set $y = xs$. We have

$$\int_0^{y_0}(x^2+y^2)^{-1}dy = x^{-1}\int_0^{y_0/x}(1+s^2)^{-1}ds \tag{A1.7}$$

$$= x^{-1}\int_0^{\infty}(1+s^2)^{-1}ds - x^{-1}\int_{y_0/x}^{\infty}(1+s^2)^{-1}ds.$$

This gives

$$\int_0^{y_0}(x^2+y^2)^{-1}dy = (\pi/2)\, x^{-1} + O(1), \tag{A1.8}$$

and also we have

$$\text{(A1.9)} \qquad \left|\int_0^{y_0} O(y^2)(x^2+y^2)^{-1}dy\right| \text{ is bounded by a constant.}$$

Substituting (A1.8) and (A1.9) into (A1.6), gives

$$\text{(A1.10)} \qquad I_1(z) = \left[1 + O(x^2)\right]\left[i\pi + O(x)\right]$$

$$= i\pi + O(x)$$

$$= i\pi + O(|z^{1/2}|).$$

To obtain the asymptotic form for $I_2(z)$ we use

$$\text{(A1.11)} \qquad \log t = i\pi + \log|t|,$$

which leads to

$$\text{(A1.12)} \quad I_2(z) = \left[1 + O(x^2)\right]2ix\int_0^{y_0}\left[1 + O(y^2)\right](x^2+y^2)^{-1}\left[i\pi + \log(y^2)\right]dy.$$

Now consider the integral

$$\text{(A1.13)} \qquad I_3(z) = x\int_0^{y_0} \log(y^2)(x^2+y^2)^{-1}dy.$$

Using $y = xs$ again, we have

$$\text{(A1.14)} \qquad I_3(z) = 2\int_0^{y_0/x} (\log x + \log s)(1+s^2)^{-1}ds,$$

which leads to

$$\text{(A1.15)} \qquad I_3(z)/2 = \int_0^{y_0/x} \log x(1+s^2)^{-1}ds$$

$$+ \int_0^{\infty} \log s(1+s^2)^{-1}ds - \int_{y_0/x}^{\infty} \log s(1+s^2)^{-1}ds$$

$$= (\pi\log x)/2 + 0 + O(x\log x),$$

and this gives

$$\text{(A1.16)} \qquad I_3(z) = (\pi\log x)/2 + O(x\log x).$$

The result (A1.2) follows on applying (A1.1) to the parts of (A1.12) not of the form of $I_3(z)$. ∎

FIGURE 8.1

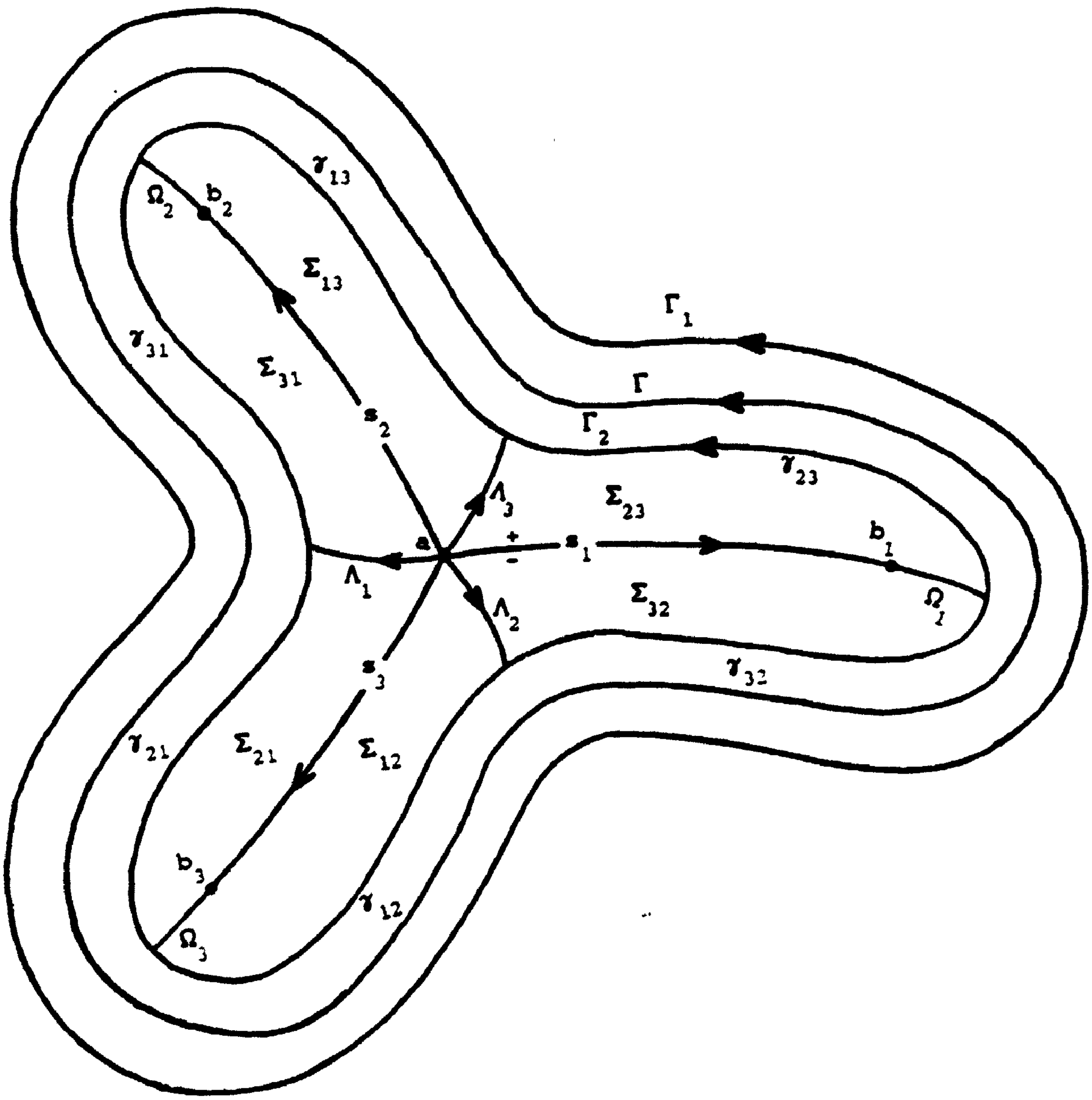

Figure 8.1: The set S_3 joining b_1, b_2, b_3 to a, and various curves used in the analysis of Padé polynomial asymptotics.

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